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# On some inverse properties for univalent functions

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*Abstract.* The object of the present paper is to investigate some inverse properties for univalent functions in the open unit disk  $U$ . Starlikeness and convexity for functions in  $U$  are shown.

## 1 Introduction

Let  $A$  denote the class of functions  $f(z)$  of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ . Let  $S$  be the subclass of  $A$  consisting of functions  $f(z)$  which are univalent in  $U$ . It is very famous as Bieberbach conjecture that if  $f(z) \in S$ , then

$$|a_n| \leq n \quad (n = 2, 3, 4, \dots). \quad (1.2)$$

The equality holds true for the Koebe function  $k(z)$  which given by

$$k(z) = \frac{z}{(1 - e^{i\theta} z)^2} \quad (\theta \in \mathbb{R}). \quad (1.3)$$

This Bieberbach conjecture was proved by de Branges [1].

In the present paper, we investigate some inverse properties for functions  $f(z)$  belonging to the class  $S$ .

Let  $B$  denote the class of functions  $f(z)$  of the form (1.1) which satisfy the coefficient inequalities (1.2). Recently, Kim and Nunokawa [2, Theorem 1] proved that if  $f(z) \in B$ , then  $f(z)$  is univalent in  $|z| < r_0$ , where  $r_0$  is the unique solution of the equation

$$2r^3 - 6r^2 + 7r - 1 = 0. \quad (1.4)$$

This result is sharp.

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## 2 Inverse properties

For the functions  $f(z)$  belonging to the class  $B$ , we derive

**Theorem 1.** *If  $f(z) \in B$ , then*

$$\frac{2r^2 - 4r + 1}{(1 - r)^2} \leq \left| \frac{f(z)}{z} \right| \leq \frac{1}{(1 - r)^2} \quad (2.1)$$

for  $|z| = r < 1$ . The result is sharp for  $f(z) = z/(1 - e^{i\theta}z)^2$ .

*Proof.* Since  $f(z) \in B$  satisfies (1.2), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} n |z|^n = \frac{r}{(1 - r)^2} \end{aligned} \quad (2.2)$$

for  $|z| = r < 1$ .

Therefore,  $f(z)$  absolutely converges in  $U$ , and so,  $f(z)$  is analytic in  $U$ .

On the other hand, we have

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\geq r - \sum_{n=2}^{\infty} nr^n \geq \frac{(2r^2 - 4r + 1)r}{(1 - r)^2} \end{aligned} \quad (2.3)$$

for  $|z| = r < 1$ . □

**Remark 1.** Theorem 1 shows that  $|f(z)/z| > 0$  for  $|z| < r_1 = \frac{2-\sqrt{2}}{2} \doteq 0.29289$ . Thus Theorem 1 is sharp.

Next we show

**Theorem 2.** *If  $f(z) \in B$ , then  $f(z)$  is univalent and starlike in  $|z| < r_2$ , where*

$$r_2 = \frac{1}{1 + \sqrt{2}} \left( 1 - \sqrt{\frac{e}{2e - 1}} \right) \doteq 0.08998. \quad (2.4)$$

*Proof.* By means of Theorem 1, we have  $|f(z)/z| > 0$  in  $|z| < r_1 = (2 - \sqrt{2})/2$ , and therefore,  $\log(f(z)/z)$  is harmonic in  $|z| < r_1$ .

From the harmonic function theory, we know that

$$\log \frac{f(z)}{z} = \frac{1}{2\pi} \int_0^{2\pi} \left( \log \left| \frac{f(\zeta)}{\zeta} \right| \right) \frac{\zeta + z}{\zeta - z} d\varphi, \quad (2.5)$$

where  $\zeta = \rho e^{i\varphi}$  ( $0 \leq \varphi \leq 2\pi$ ),  $z = r e^{i\theta}$  ( $0 \leq \theta \leq 2\pi$ ), and  $0 \leq r < \rho \leq r_1 = (2 - \sqrt{2})/2$ .

By using the logarithmic differentiation, we obtain

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2\pi} \int_0^{2\pi} \left( \log \left| \frac{f(\zeta)}{\zeta} \right| \right) \frac{2\zeta z}{(\zeta - z)^2} d\varphi. \quad (2.6)$$

Because, we have

$$\frac{1}{(1-r)^2} < \frac{(1-r)^2}{2r^2 - 4r + 1} \quad (2.7)$$

for  $|z| = r < 1$ , then, from Theorem 1 and (2.7), we derive

$$\begin{aligned} \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} &\geq 1 - \frac{1}{2\pi} \int_0^{2\pi} \left( \max_{|\zeta|=\rho} \left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \right) \frac{2\rho r}{\rho^2 - 2\rho \cos(\varphi - \theta) + r^2} d\varphi \\ &\geq 1 - \frac{2\rho r}{\rho^2 - r^2} \log \frac{(1-\rho)^2}{2\rho^2 - 4\rho + 1}, \end{aligned} \quad (2.8)$$

where  $0 \leq r < \rho < r_1 = (2 - \sqrt{2})/2$ .

Putting  $\rho = (1 + \sqrt{2})r$ , we have

$$\frac{2\rho r}{\rho^2 - r^2} \log \left( \frac{\rho^2 - 2\rho + 1}{2\rho^2 - 4\rho + 1} \right) = \log \left( \frac{1}{2} + \frac{1}{4 \{ (1 + \sqrt{2})r - 1 \}^2 - 2} \right) = 1. \quad (2.9)$$

Consequently, we see that (2.8) and (2.9) imply

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (2.10)$$

in  $|z| < r_2$ , where  $r_2$  is the smallest positive root of the equation

$$\frac{1}{2} + \frac{1}{4 \{ (1 + \sqrt{2})r - 1 \}^2 - 2} = e \quad (2.11)$$

or

$$r_2 = \frac{1}{1 + \sqrt{2}} \left( 1 - \sqrt{\frac{e}{2e-1}} \right) \approx 0.08998. \quad (2.12)$$

This completes the proof of Theorem 2.  $\square$

**Remark 2.** In the proof of Theorem 2, we put  $\rho = (1 + \sqrt{2})r$ . But we don't prove that this is best or not. Therefore, Theorem 2 is not sharp.

From Theorem 2, we make

**Corollary 1.** *If a function  $f(z)$  of the form (1.1) satisfies*

$$|a_n| \leq 1 \quad (n = 2, 3, 4, \dots),$$

*then  $f(z)$  is univalent and convex in  $|z| < r_2$ .*

Applying the same method as the proof of Theorem 2, we can obtain some rough results on the other cases, but we expect that someone get exact results.

## References

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