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Author(s)	Shiba, Masakazu
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Some function-theoretic properties of the hyperbolic metric and a new construction of fundamental domains

M. Shiba, Hiroshima University
(柴　雅和, 広島大) [1] [2]

§0. Introduction.

In this article two relationships between the hyperbolic metric and the Riemann surface theory are studied. After preparing some results of the author ([3]) which motivate the subsequent observation and discussion, we give a short report of the author's two joint papers ([1] and [2]) with Masumoto.

The first relationship is rather indirect. The hyperbolic metric is connected with the area of the complement of the conformal image of a fixed plane domain. Such a property is shared with also by conformally embedded noncompact Riemann surface of genus one in various tori. In fact, we had found the property first for the case of genus one, and then knew that the same is valid for the classical case. As an application of our result, we can also show a simple yet general relation between the hyperbolic, euclidean, and spherical metrics. See [3]. The results in [3] have been generalized to general Riemann surfaces in [1], where the proofs are slightly simplified.

The second relationship is, on the other hand, direct. Being motivated by the previous works [3] and [1], we observe the hyperbolic metric on various simply connected domains on a fixed Riemann surface and consider an extremal problem. We see that the extremal problem has a unique solution and show that the extremal domain has very natural and interesting properties, which are described in terms of quadratic differentials. It is worth while to point out that the extremality is closely connected with the Riemann mapping theorem.

Finally, as an application of the above results, we give a method how to construct a fundamental domain for a discontinuous group consisting of

[1]The author thanks Prof. Kamiya for his kind invitation to the conference.
[2]E-mail: shiba@amath.hiroshima-u.ac.jp

conformal automorphisms of the Riemann surface under consideration. We have, in particular, a new method of construction of a fundamental domain for a Fuchsian group. The fundamental domain thus obtained is in general neither Dirichlet nor Ford.

§1. Conformal embedding of an open torus into tori and area theorems.

We begin with some basic results necessary for the subsequent sections. For the full-length discussion, see [3].

Let R be a noncompact Riemann surface of genus one, which will be called "an open torus" for short. Fix a canonical homology basis $\chi = \{a, b\}$ of R modulo dividing cycles. The pair (R', χ') will be called a (homologically) marked open torus.

Let R' be a torus (= a compact Riemann surface of genus one) and $\chi' = \{a', b'\}$ a canonical homology basis of R'. We can and do always assume that the curve a' is a geodesic with respect to the naturally defined flat metric on the torus R' which comes from an (essentially unique) holomorphic differentail on R'.

If a conformal mapping i' of R into R' preserves the homology, i.e., if

$$i'(a) \sim a', \quad i'(b) \sim b'$$

(where $\sim$ reads "is homologous to"), then we say that i' is a conformal embedding of (R, χ) into (R', χ') and simply write as

$$i' : (R, \chi) \to (R', \chi'), \quad \text{conformal.}$$

Two conformal embeddings

$$i' : (R, \chi) \to (R', \chi'), \qquad i'' : (R, \chi) \to (R'', \chi'')$$

are said to be equivalent if there exists a conformal (necesssarily surjective) mapping $f : R' \to R''$ with $f \circ i' = i''$. Each equivalence class $[R', \chi', i']$ is called a (compact) continuation of (R, χ), and the set of all continuations of (R, χ) is denoted by $C(R, \chi)$. The modulus of a $[R', \chi', i'] \in C(R, \chi)$ is, by definition, the modulus of the marked torus (R', χ'), and is denoted by $\tau[R', \chi', i']$. The set of moduli $\tau[R', \chi', i'] \in C(R, \chi)$ is denoted by

$$M(R, \chi) = \{\tau \in \mathbb{C} \mid \tau = \tau[R', \chi', i'], \ [R', \chi', i'] \in C(R, \chi)\}.$$

It is a subset of the (open) upper half plane

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}.$$

The following theorem describes the set $C(R,\chi)$ or $M(R,\chi)$ in the euclidian geometry. Part (2) and (3) are not always necessary for our discussion below, which are included here for the later reference (see §2), however.

THEOREM. (1) $M(R,\chi)$ is a closed disk: $|\tau - \tau_E| \leq \rho_E$, $\operatorname{Im} \tau_E > 0$, $0 \leq \rho_E < \operatorname{Im} \tau_E$.
(2) Parametrize $\partial M(R,\chi)$ as

$$\tau_t = \tau_E + \rho_E e^{(t-\frac{1}{2})\pi i} \; (-1 < t \leq 1).$$

Then, to each point $\tau_t \in \partial M(R,\chi)$ there corresponds a unique continuation $[R_t, \chi_t, i_t] \in C(R,\chi)$ such that the complement $R_t \setminus i_t(R)$ is a Lebesgue null set consisting of parallel segments whose inclination with the geodesic a_t is $\pi t/2$, where $\chi_t = \{a_t, b_t\}$.
(3) $\rho_E = 0 \Leftrightarrow R \in O_{AD}$. Here O_{AD} stands for the class of Riemann surfaces which carry no nonconstant (singlevalued) analytic functions with a finite Dirichlet integral.

To investigate more detailed properties of the moduli disk $M(R,\chi)$ and to see its close relation with the classical theory of univalent functions, we consider the continuations with a fixed modulus τ:

$$C_\tau(R,\chi) := \{[R',\chi',i'] \in C(R,\chi) \mid \tau[R',\chi',i'] = \tau\}, \qquad \tau \in M(R,\chi).$$

In other words, we consider all possible conformal embeddings of (R,χ) into a fixed marked torus (R',χ') with modulus τ.

The euclidean disk $M(R,\chi)$ is, as is well known, simultaneously a hyperbolic disk in $\mathbb{H}$ as well as a spherical disk in the Riemann sphere $\hat{\mathbb{C}}$. The hyperbolic and spherical centers of $M(R,\chi)$ are respectively denoted by τ_H and τ_S, and similarly for the radii: the hyperbolic radius of $M(R,\chi)$ is ρ_H and the spherical radius ρ_S.

The domains $\mathbb{H}$, $\mathbb{C}$, and $\hat{\mathbb{C}}$ have standard metrics with their curvature normalized. Denote by

$$ds_{\mathbb{H}}, \; ds_{\mathbb{C}}, \; \text{and} \; ds_{\hat{\mathbb{C}}}$$

the normalized hyperbolic, euclidean, and spherical metrics on

$$\mathbb{H}, \quad \mathbb{C}, \quad \text{and } \hat{\mathbb{C}}$$

respectively. By the normalization we can write them as

$$ds_{\mathbb{H}} = \frac{|d\tau|}{\operatorname{Im}\tau}, \quad ds_{\mathbb{C}} = |d\tau|, \quad \text{and } ds_{\hat{\mathbb{C}}} = \frac{2|d\tau|}{1+|\tau|^2}.$$

The distance functions induced from these metrics are respectively denoted by

$$d_{\mathbb{H}}(\cdot,\cdot), \quad d_{\mathbb{C}}(\cdot,\cdot), \quad \text{and } d_{\hat{\mathbb{C}}}(\cdot,\cdot).$$

We finally set

$$r_H := d_{\mathbb{H}}(\tau, \tau_H), \quad r_E := d_{\mathbb{C}}(\tau, \tau_E), \quad \text{and } r_S := d_{\hat{\mathbb{C}}}(\tau, \tau_S),$$

where τ is a point in $M(R,\chi)$.

Each marked torus (R', χ') carries a unique normalized holomorphic differential ϕ such that

$$\int_{a'} \phi = 1,$$

which induces the standard euclidean metric on R'. The area of the whole surface R' measured by this metric is denoted by

$$A[R'] = A[R', \chi', i']$$

and the area of the complement of the embedded surface $i'(R)$ in R' similarly measured is denoted by

$$\alpha[R'] = \alpha[R', \chi', i'].$$

Furthermore, we consider

$$\alpha_\tau := \sup\{\alpha[R', \chi', i'] \mid [R', \chi', i'] \in C_\tau(R,\chi)\}, \qquad \tau \in M(R,\chi).$$

With these definitions we have

THEOREM. (1) There exists a unique $[R', \chi', i'] \in C_\tau(R,\chi)$ with $\alpha[R', \chi', i'] = \alpha_\tau$.

(2) $\alpha_\tau = \dfrac{\rho_E^2 - r^2}{2\rho_E} = \dfrac{|d\tau|}{ds_M}.$

(3) $\max\{\alpha[R'] \mid [R',\chi',i'] \in C(R,\chi)\} = \alpha_{\tau_E} = \dfrac{\rho_E}{2}.$

Concerning the spherical and hyperbolic metrics, we first set

$$S[R'] := \frac{\alpha[R']}{A[R']}, \qquad S_\tau := \sup\{S[R',\chi',i'] \mid [R',\chi',i'] \in C_\tau(R,\chi)\}$$

$$\Delta[R'] := \frac{\alpha[R']}{\pi(1+|\tau|^2)}, \qquad \Delta_\tau := \sup\{\Delta[R',\chi',i'] \mid [R',\chi',i'] \in C_\tau(R,\chi)\}$$

and have the following two theorems:

THEOREM. (1) There exists a unique $[R',\chi',i'] \in C_\tau(R,\chi)$ with $S[R',\chi',i'] = S_\tau$.

(2) $S_\tau = \dfrac{\cosh \rho_H - \cosh r}{\sinh \rho_H}.$

(3) $\max\{S[R',\chi',i'] \mid [R',\chi',i'] \in C(R,\chi)\} = S_{\tau_H} = \tanh\dfrac{\rho_H}{2}.$

THEOREM. (1) There is a unique $[R',\chi',i'] \in C_\tau(R,\chi)$ with $\Delta[R',\chi',i'] = \Delta_\tau$.

(2) $\Delta_\tau = \dfrac{\tan^2 \frac{\rho_S}{2} - \tan^2 \frac{r}{2}}{\tan \frac{\rho_S}{2} \cdot \left(1 + \tan^2 \frac{r}{2}\right)}.$

(3) $\max\{S[R',\chi',i'] \mid [R',\chi',i'] \in C(R,\chi)\} = \Delta_{\tau_S} = \tan\dfrac{\rho_S}{2}.$

§2. Area theorems in the classical theory of univalent functions.

To state the prototype of the theorems in the preceding section and to state some new results in the classical theory, we now consider a general domain $G\,(\subset \hat{\mathbb{C}})$ and fix a point $\zeta \in G$. For simplicity we assume that $\zeta = \infty$. Let $F(G,\zeta)$ be the class of all conformal embeddings of G into $\hat{\mathbb{C}}$ normalized at ζ. That is, we set

$$F(G,\zeta) := \left\{ f : G \to \hat{\mathbb{C}} \;\middle|\; \begin{array}{l} f \text{ is univalent meromorphic and} \\ f(z) = 1/(z-\zeta) + \kappa_f(z-\zeta) + \cdots \text{ about } \zeta \end{array} \right\}.$$

Furthermore we consider the set of coefficients of $(z-\zeta)$:

$$K(G,\zeta) := \{\kappa \in \mathbb{C} \mid \kappa = \kappa_f \text{ for some } f \in F(G,\zeta).$$

THEOREM. (1) $F(G,\zeta)$ is a closed disk: $|\, \kappa - \kappa^* \,| \leq \rho^*, \quad \kappa^* \in \mathbb{C},\ \rho^* \geq 0.$
(2) Parametrize the periphery $\partial K(G,\zeta)$ of the disk as

$$\kappa_t = \kappa^* + \rho^* e^{t\pi i} \ (-1 < t \leq 1).$$

Then, to each point $\kappa_t \in \partial K(G,\zeta)$ there corresponds a unique $f_t \in F(G,\zeta)$ such that $\hat{\mathbb{C}} \backslash f_t(G)$ is a null set consisting of parallel segments with inclination $\pi t/2$.
(3) $\rho^* = 0 \Leftrightarrow G \in O_{AD}$.

§3. Hyperbolic metric and other metrics.

We write each of the pairs

$$(\mathbb{H}, ds_{\mathbb{H}}), \quad (\mathbb{C}, ds_{\mathbb{C}}), \quad \text{and } (\hat{\mathbb{C}}, ds_{\hat{\mathbb{C}}})$$

simply as (X, ds_X). Let D be a disk in $\mathbb{H}$ and ds_D the hyperbolic metric of D itself. From the results obtained in §1 and §2, we easily have:

THEOREM. For each $X = \mathbb{H}$, or $\mathbb{C}$, or $\hat{\mathbb{C}}$, the differential quotient

$$\frac{ds_X}{ds_D}$$

assumes a constant value on each concentric circle of (X, ds_X).

We invoke the invariance of the hyperbolic metric under Möbius transformations, to implant the above theorem into Riemann surface theory. To this end we need the following definition.

DEFINITION. Let R be a general Riemann surface, d the distance function induced from the complete conformal metric on R with constant curvature. A simply connected domain D on R is called an *open intrinsic disk* with center at $p_0 \in R$ if
(i) ∂D includes more than one point, and
(ii) D is given by

$$D = \{p \in R \mid d(p, p_0) < r\}$$

for some positive r. If the closure $\bar{D}$ of an open intrinsic disk D is again simply connected, then we call $\bar{D}$ a *closed intrinsic disk*. The boundary of a closed intrinsic disk is always a closed Jordan curve [3], which we call an *intrinsic circle.*

The following theorem is a generalization of the previous theorem.

THEOREM. Let R be a Riemann surface and D an open intrinsic disk with center at p_0. Denote by ds_R^2 and ds_D^2 the complete conformal metrics on R and D with constant curvature, respectively. Then, the differential quotient

$$\frac{ds_R}{ds_D}(p)$$

is a smooth function of $r := d_R(p, p_0)$, and in particular, it assumes a constant value on each concentric intrinsic circle.

§4. Hyperbolic maximal domains and fundamental domains for a discrete group of conformal automorphisms.

Now we recall the Riemann mapping theorem and one of its proofs. It is reasonable to consider an extremal problem with respect to the differential quotients.

Let R be a Riemann surface and let Γ be a discrete subgroup of conformal automorphisms of R. Denote by R^* the set of points on R which are fixed by no nontrivial element of Γ. We assume, furthermore, that the quotient R^*/Γ is conformally equivalent to neither $\mathbb{C}$ nor $\hat{\mathbb{C}}$.

For any fixed point $p \in R^*$ we define $\mathscr{D}_p^\Gamma(R)$ to be the class of simply connected domains D on R such that

(i) D contains the point p, and

(ii) $\gamma(D) \cap D = \emptyset$ for any $\gamma \in \Gamma \setminus \{id.\}$.

The class $\mathscr{D}_p^\Gamma(R)$ has a quasi-order relation $\underset{p}{\preceq}$ which is defined as follows:

DEFINITION. For $D_1, D_2 \in \mathscr{D}_p^\Gamma(R)$: $D_1 \underset{p}{\preceq} D_2 \iff \dfrac{ds_{D_2}}{ds_{D_1}}(p) \leq 1.$

Roughly speaking, the class is quasi-ordered by observing whether the natural local inclusion map is a contraction.

THEOREM. Each class $\mathscr{D}_p^\Gamma(R)$, where $p \in R^*$, has a unique maximum element.

[3]This is not always the case for the boundary of an open intrinsic disk.

The following terminology is useful:

DEFINITION. Let R, Γ, and R^* be as above. A simply connected domain D on R is called *hyperbolically maximal* if it is the maximum element in the class $\mathscr{D}_p^\Gamma(R)$ for some $p \in R^*$.

A hyperbolically maximal domain has remarkable function-theoretic properties, which are studied in [2]. Here we state only some of them.

THEOREM. Let R, Γ, and R^* be as before. Let D be a hyperbolically maximal domain for Γ. Then

(i) D is a fundamental domain for Γ.

(ii) D is locally finite.

(iii) ∂D is a Lebesgue null set.

(iv) ∂D is piecewise analytic and the corner angle at a vertex is $2\pi/n$ with an integer n.

Our extremal problem thus gives a new method how to construct a fundamental domain for a Fuchsian group. By statement (iv) of the last theorem we see that the fundamental domain obtained in this way coincides with neither Ford nor Dirichlet. For the detailed discussion and examples see [2].

References

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