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## Facial structure of convex sets and some applications

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### §1 INTRODUCTION

Let  $\Omega$  be a measure space and let  $S(\Omega)$  be the space of all measurable functions  $f$  on  $\Omega$  such that  $f(t) < \infty$  (a.e.  $t \in \Omega$ ). An operator  $F : X \supset D(F) \rightarrow S(\Omega)$  is called a convex operator if  $D(F)$  is a convex set in a real vector space  $X$ , and for each  $x, y \in D(F)$  and  $0 < \alpha < 1$ ,

$$F((1 - \alpha)x + \alpha y)(t) \leq (1 - \alpha)F(x)(t) + \alpha F(y)(t) \quad (\text{a.e. } t \in \Omega).$$

On the other hand, a function  $f : X \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  is called a convex integrand if for each  $t \in \Omega$  the function  $f(\cdot, t)$  is convex on  $\mathbb{R}^d$ . The convex integrand theory is well known and there are many applications. (See [7] for example.) We say that a convex integrand  $f$  represents a convex operator  $F$  if

$$(1) \quad f(x, t) = \begin{cases} F(x)(t) & \text{for a.e. } t \in \Omega \quad x \in D(F) \\ \infty & \quad \quad \quad x \notin D(F) \end{cases}$$

In two of the author's previous paper [3, 4], many applications of integrand representations of convex operators were demonstrated. However, the existense of integrand representation is nontrivial, and it is known only in some special cases. When  $X$  is the  $d$ -dimensional Euclidian space  $\mathbb{R}^d$ , the representation theorem has been proved in [3]. In this note, we apply the theory of the faces of convex sets, and give a new method of the proof which is expected to have an advantage in extending the representation theorem to infinite dimensional cases.

### §2 FACES OF CONVEX SETS

Let  $\mathbb{R}^d$  be the  $d$ -dimensional Euclidean space. When  $x, y \in \mathbb{R}^d$  are distinct points, then the set  $[x, y] = \{(1 - t)x + ty \mid 0 \leq t \leq 1\}$  is called the closed segment between  $x$  and  $y$ . Half open segments  $(x, y]$ ,  $[x, y)$  and open segments  $(x, y)$  are defined analogously. Through this section, we fix a nonempty closed convex set  $D$  in  $\mathbb{R}^d$ . A convex subset  $C$  of  $D$  is called a face of  $D$  if

$$(2) \quad \left\{ \begin{array}{l} x, y \in D \\ (x, y) \cap C \neq \emptyset \end{array} \right\} \text{ implies } [x, y] \subset C.$$

By  $\mathfrak{F}(D)$ , we denote the set of all faces of  $D$ . For  $C \in \mathfrak{F}(D)$ ,  $\dim C$  is defined to be the dimension of  $\text{aff } C$  (the affine hull of  $C$ ). It is clear that  $x \in D$  is an extreme point of  $D$  if and only if  $\{x\}$  is a 0-dimensional face of  $D$ . For preparation, we will state some fundamental properties of faces in the following propositions whose proofs are given in [1].

**Proposition 1.** *If  $C_\lambda \in \mathfrak{F}(D)$ , ( $\lambda \in \Lambda$ ), then  $\bigcap_{\lambda \in \Lambda} C_\lambda \in \mathfrak{F}(D)$ , and also there exists a smallest face of  $D$  containing  $\bigcup_{\lambda \in \Lambda} C_\lambda$ . Hence  $(\mathfrak{F}(D), \subset)$  forms a complete lattice.*

**Proposition 2.** *Let  $C_1$  be a face of  $D$  and suppose that  $C_2 \subset C_1$ . Then  $C_2 \in \mathfrak{F}(D)$  if and only if  $C_2 \in \mathfrak{F}(C_1)$ .*

For a convex set  $C$  in  $\mathbb{R}^d$ ,  $\overset{\circ}{C}$  denotes the relative interior of  $C$ , which means the interior of  $C$  with respect to the relative topology of  $\text{aff } C$ . It is easy to see that every face of  $D$  is a closed set. Indeed, if  $x$  is a point of the closure of a face  $C$  and  $x_0 \in \overset{\circ}{C}$ , the convexity of  $C$  yields  $[x_0, x) \subset \overset{\circ}{C} \subset C$ . Since  $C$  is a face of  $D$ ,  $x$  must be in  $C$ .

**Proposition 3.** *If  $C_1, C_2 \in \mathfrak{F}(D)$ , and  $C_1 \not\subseteq C_2$ , then  $C_1 \cap \overset{\circ}{C_2} = \emptyset$ .*

**Proposition 4.** *Let  $x$  be a point of  $D$  and let  $C$  be a face of  $D$ . Then  $C$  is the smallest face of  $D$  containing  $x$  if and only if  $x \in \overset{\circ}{C}$ .*

**Proposition 5.** *Let  $C_1$  be a face of  $D$  and let  $x$  be a relative boundary point of  $C_1$ . If  $C_2$  is the smallest face of  $D$  containing  $x$ , then  $C_2$  is contained by the relative boundary of  $C_1$ .*

From these propositions we obtain the following decomposition of a convex set by its faces.

**Proposition 6.** *For a closed convex set  $D$  in  $\mathbb{R}^d$ ,*

$$D = \bigcup \{ \overset{\circ}{C}_\lambda \mid C_\lambda \in \mathfrak{F}(D) \}$$

and the union is disjoint.

We say that a collection  $\{C_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{F}(D)$  is normal if  $\lambda \in \Lambda$  and  $C_\lambda \subset C_\mu \in \mathfrak{F}(D)$  imply  $\mu \in \Lambda$ . Now we define

$$\mathfrak{A} = \left\{ A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda \mid \{C_\lambda\}_{\lambda \in \Lambda} \text{ is normal} \right\}.$$

Since  $\{\overset{\circ}{D}\}$  is normal and  $\overset{\circ}{D} \in \mathfrak{A}$ ,  $\mathfrak{A}$  is at least nonempty. It is easy to see that if each  $A_\lambda$  ( $\lambda \in \Lambda$ ) is a member of  $\mathfrak{A}$ , then so are  $\bigcup_{\lambda \in \Lambda} A_\lambda$  and  $\bigcap_{\lambda \in \Lambda} A_\lambda$ , and therefore  $(\mathfrak{A}, \subset)$  is a complete lattice.

**Lemma 1.** *If  $A \in \mathfrak{A}$ , then  $A$  is a convex set.*

*proof.* We write  $A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda$  and let  $x, y$  be arbitrary points of  $A$ . Then there exist  $\lambda$  and  $\mu$  such that  $x \in \overset{\circ}{C}_\lambda$  and  $y \in \overset{\circ}{C}_\mu$ . Let  $z$  be an arbitrary point of the open segment  $(x, y)$ , and let  $C_\nu$  be the smallest face containing  $z$ . Since  $C_\nu$  is a face, we have  $[x, y] \subset C_\nu$ . By Proposition 4,  $C_\lambda$  is the smallest face containing  $x$ , and it follows that  $C_\lambda \subset C_\nu$ . Since the collection  $\{C_\lambda\}_{\lambda \in \Lambda}$  is normal, we obtain  $\overset{\circ}{C}_\nu \subset A$ . This means that  $z \in A$ , and thus  $A$  is convex.

### §3 REPRESENTATION OF CONVEX OPERATORS

In this section, we prove a representation theorem of convex operators. Let  $D(F)$  be a convex set in  $\mathbb{R}^d$  and let  $F : D(F) \rightarrow S(\Omega)$  be a convex operator. We can assume without loosing generality that the interior of  $D(F)$  is nonempty. Through this section,  $\overline{D}$  denotes the closure of  $D(F)$ . First we state the main theorem.

**Theorem 1.** *Every convex operator  $F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)$  has at least a representation. That is, there exists a convex integrand  $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  such that (1) holds.*

For  $D = \overline{D(F)}$ , we define  $\mathfrak{A}$  as in §2. For  $A \in \mathfrak{A}$ , a convex integrand  $f : A \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  is said to represent  $F$  on  $A$ , if

$$f(x, t) = \begin{cases} F(x)(t) & \text{for a.e. } t \in \Omega & x \in A \cap D(F) \\ \infty & & x \in A \setminus D(F). \end{cases}$$

**Definition.** For a convex operator  $F$ , we define

$$\tilde{\mathfrak{A}} = \{(A, f) \mid A \in \mathfrak{A}, \text{ and } f \text{ represents } F \text{ on } A\}.$$

Moreover, for  $(A_1, f_1), (A_2, f_2) \in \tilde{\mathfrak{A}}$ , we write  $(A_1, f_1) \leq (A_2, f_2)$  when  $A_1 \subset A_2$  and  $f_2$  is an extension of  $f_1$  to  $A_2$ .

**Lemma 2.**  $(\tilde{\mathfrak{A}}, \leq)$  is inductively ordered.

*proof.* Let  $\{(A_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$  be a totally ordered subset of  $\tilde{\mathfrak{A}}$ . Then  $A = \bigcup_{\lambda \in \Lambda} A_\lambda$  is an element of  $\mathfrak{A}$ . Moreover we can define a convex integrand  $f$  on  $A \times \Omega$  satisfying  $f = f_\lambda$  on  $A_\lambda \times \Omega$  for every  $\lambda \in \Lambda$ . Clearly,  $(A, f) \in \tilde{\mathfrak{A}}$  and it is an upper bound of  $\{(A_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$ .

**Lemma 3.** For  $A \in \mathfrak{A}$  such that  $A \neq D$ , we define  $\mathfrak{S}_A = \{C \in \mathfrak{F}(D) \mid C \cap A = \emptyset\}$ . Then  $(\mathfrak{S}_A, \subset)$  is inductively ordered.

*proof.* Let  $\{C_\lambda\}_{\lambda \in \Lambda}$  be a totally ordered subset of  $\mathfrak{S}_A$ . If we put  $C = \bigcup_{\lambda \in \Lambda} C_\lambda$ , then  $C$  is a convex set and  $C \cap A = \emptyset$ . Moreover  $C \in \mathfrak{F}(D)$ . Indeed, if we assume  $(x, y) \cap C \neq \emptyset$ , then there exists  $\lambda \in \Lambda$  such that  $(x, y) \cap C_\lambda \neq \emptyset$ . Hence it follows that  $[x, y] \subset C_\lambda \subset C$ . Thus  $C \in \mathfrak{S}_A$  and it is an upper bound of  $\{C_\lambda\}_{\lambda \in \Lambda}$ .

**Lemma 4.** Let  $A$  be an element of  $\mathfrak{A}$ , and assume that  $A \neq D$ . Then there exists  $C \in \mathfrak{S}_A$  such that  $A \cup \overset{\circ}{C} \in \mathfrak{A}$ .

*proof.* By Lemma 3 and Zorn's lemma,  $\mathfrak{S}_A$  has at least a maximal element  $C$ . It is sufficient to show that  $A \cup \overset{\circ}{C} \in \mathfrak{A}$ . Put  $A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda$ , and take  $C_1 \in \mathfrak{F}(D)$ , such that  $C_1 \supset C$ . Since  $C$  is a maximal element of  $\mathfrak{S}_A$ , we have  $C_1 \notin \mathfrak{S}_A$  and hence  $C_1 \cap A \neq \emptyset$ . Therefore we can choose  $\lambda \in \Lambda$  such that  $\overset{\circ}{C}_\lambda \cap C_1 \neq \emptyset$ . It follows from Proposition 3 that,  $C_\lambda \subset C_1$  holds. Since the collection  $\{C_\lambda\}_{\lambda \in \Lambda}$  is normal,  $\overset{\circ}{C}_1 \subset A \subset A \cup \overset{\circ}{C}$ . This shows that the collection  $\{C_\lambda\}_{\lambda \in \Lambda} \cup \{C\}$  is also normal, and  $A \cup \overset{\circ}{C} \in \mathfrak{A}$ .

**Lemma 5.**  $\tilde{\mathfrak{A}}$  is not empty. In other words, there exists  $A \in \mathfrak{A}$  such that  $F$  has a representation  $f$  on  $A$ .

The proof can be done by constructing a convex integrand  $f$  which represents  $F$  on  $\overset{\circ}{D}$ . The method of construction is an analogy of that in [4].

**Lemma 6.** *Suppose that  $(A, f) \in \tilde{\mathfrak{A}}$  and  $A \neq D$ . Let  $C \in \mathfrak{S}_A$  is a face such that  $A \cup \overset{\circ}{C} \in \mathfrak{A}$  as in Lemma 4. Then  $f$  has an extension  $f_1$  defined on  $(A \cup \overset{\circ}{C}) \times \Omega$  such that  $(A \cup \overset{\circ}{C}, f_1) \in \tilde{\mathfrak{A}}$ .*

The proof of this lemma is an analogy of one provide in a previous paper by the author [3].

*proof of Theorem 1.* By Lemma 3, Lemma 5 and Zorn's lemma,  $\tilde{\mathfrak{A}}$  has at least a maximal element  $(A_0, f_0)$ . Moreover, Lemma 6 shows that  $A_0 = D$ , and this means that  $f_0$  represents  $F$  on  $D$ . Defining  $f_0 = \infty$  on  $D^c \times \Omega$ , we complete the construction of a representation of  $F$ .

#### §4 NORMAL REPRESENTATIONS

A convex integrand  $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  is said to be normal if  $f(\cdot, t)$  is lower semicontinuous for every  $t \in \Omega$  and there exists a countable family of measurable functions  $\xi_n : \Omega \rightarrow \mathbb{R}^d$  ( $n = 1, 2, \dots$ ) such that

(1) for each  $n$ ,  $f(\xi_n(t), t)$  is measurable in  $t \in \Omega$ ,

(2) for each  $t \in \Omega$ ,  $\{\xi_n(t)\}_{n=1}^{\infty}$  is dense in  $D(f(\cdot, t))$ ,

where  $D(f(\cdot, t)) = \{x \in \mathbb{R}^d \mid f(x, t) < \infty\}$ . If a convex integrand  $f$  is normal, then  $f(\xi(t), t)$  is measurable in  $t \in \Omega$  whenever  $\xi : \Omega \rightarrow \mathbb{R}^d$  is measurable. A convex operator  $F$  is said to have a normal representation if there exists a normal convex integrand which represents  $F$ . We will find some conditions under which a convex operator has a normal representation. By the conjugate of a convex integrand  $f$ , we mean the convex integrand  $f^* : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$f^*(\xi, t) = \sup_{x \in \mathbb{R}^d} \{ \langle x, \xi \rangle - f(x, t) \}.$$

Also the biconjugate  $f^{**} : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \infty$  is given by

$$f^{**}(x, t) = \sup_{\xi \in \mathbb{R}^d} \{ \langle x, \xi \rangle - f^*(\xi, t) \}.$$

If a convex integrand  $f$  is normal, then so are  $f^*$  and  $f^{**}$ . We note that if a convex integrand  $f$  represents a convex operator  $F$  then  $D(f(\cdot, t))$  does not depend on  $t \in \Omega$ .

**Lemma 7.** Let  $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  be a representation of some convex operator. Then  $f$  is normal if and only if  $f(\cdot, t)$  is lower semicontinuous, in other words,  $f^{**} = f$  on  $\mathbb{R}^d \times \Omega$ .

*proof.* Let  $D = D(f(\cdot, t))$  and take a countable subset  $\{a_n\}$  of  $D$ . If we put  $\xi_n(t) = a_n$  for all  $t \in \Omega$  and  $n = 1, 2, \dots$ , then the family  $\{\xi_n\}$  satisfies the definition of normality.

*Remark.* If a convex integrand  $f$  satisfies

- (1) for each  $x \in \mathbb{R}^d$ ,  $f(x, \cdot)$  is measurable, and
- (2)  $\overline{D(\cdot, t)}$  does not depend on  $t \in \Omega$ ,

the conclusion of Lemma 7 is also valid.

Let  $L(\mathbb{R}^d, S(\Omega))$  denotes the space of all linear mapping from  $\mathbb{R}^d$  to  $S(\Omega)$ . We identify  $L(\mathbb{R}^d, S(\Omega))$  with the set  $S(\Omega)^d = \{\xi = (\xi_1, \dots, \xi_d) \mid \xi_i \in \xi(\Omega), i = 1, \dots, d\}$  by corresponding  $S(\Omega)^d \ni (\xi_1, \dots, \xi_d)$  to the mapping  $\varphi : \mathbb{R}^d \ni (x_1, \dots, x_d) \rightarrow \langle x, \xi \rangle = x_1\xi_1 + \dots + x_d\xi_d \in S(\Omega)$ . The conjugate operator  $F^* : L(\mathbb{R}^d, S(\Omega)) \supset D(F^*) \rightarrow S(\Omega)$  of  $F$  is defined by

$$F^*(\xi) = \bigvee_{x \in D(F^*)} (\langle x, \xi \rangle - F(x))$$

where  $\bigvee$  means the lattice supremum in the space  $S(\Omega)$ , and  $D(F^*)$  is the set of all  $\xi \in S(\Omega)^d$  such that the supremum  $F^*$  exists. The bi-conjugate operator  $F^{**}$  is defined on the space  $L(L(\mathbb{R}^d, S(\Omega)), S(\Omega)) = L(S(\Omega)^d, S(\Omega))$ , and we regard  $S(\Omega)^d$  and  $\mathbb{R}^d$  as the subspaces of this by corresponding  $\eta \in S(\Omega)^d$  and  $x \in \mathbb{R}^d$  to  $\langle \eta, \cdot \rangle$  and  $\langle x, \cdot \rangle \in L(S(\Omega)^d, S(\Omega))$  respectively. For  $x \in \mathbb{R}^d$  and  $\eta \in S(\Omega)$ ,  $F^{**}$  is defined by

$$F^{**}(x) = \bigvee_{\xi \in D(F^*)} (\langle x, \xi \rangle - F^*(\xi)),$$

$$F^{**}(\eta) = \bigvee_{\xi \in D(F^*)} (\langle \eta, \xi \rangle - F^*(\xi)).$$

They are only defined on the domain  $D(F^{**})$  where these suprema exist.

**Theorem 2.** Let  $F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)$  be a convex operator and let  $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  be a representation of  $F$ . Then the convex integrand  $f^*$  and  $f^{**}$  are normal representations of  $F^*$  and  $F^{**}$  respectively. Moreover for  $\xi \in D(F^*)$  and  $\eta \in D(F^{**})$ ,

$$(F^*(\xi))(t) = f^*(\xi(t), t)$$

$$(F^{**}(\eta))(t) = f^{**}(\eta(t), t)$$

holds for almost every  $t \in \Omega$ .

This theorem is due to the following lemma.

**Lemma 8.** *Let  $F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)$  be a convex operator, and let  $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d \cup \{\infty\}$  be a representation of  $F$ . Let  $U$  be a convex subset of  $D(F)$  and suppose that  $\inf_{x \in U} f(x, t) > -\infty$  for almost every  $t \in \Omega$ . Then  $\bigwedge_{x \in U} F(x) \in S(\Omega)$  exists and*

$$\left( \bigwedge_{x \in U} F(x) \right)(t) = \inf_{x \in U} f(x, t).$$

*proof.* Let  $E$  be a countable dense set in  $U$ . Then we have

$$\inf_{x \in U} f(x, t) = \inf_{x \in E} f(x, t)$$

for a.e.  $t \in \Omega$ . Hence  $\inf_{x \in U} f(x, t)$  is measurable in  $t$  and

$$\begin{aligned} \left( \bigwedge_{x \in U} F(x) \right)(t) &\leq \left( \bigwedge_{x \in E} F(x) \right)(t) \\ &= \inf_{x \in E} f(x, t) \\ &= \inf_{x \in U} f(x, t) \\ &= \left( \bigwedge_{x \in U} F(x) \right)(t) \end{aligned}$$

for a.e.  $t \in \Omega$ , and the lemma is proved.

*proof of Theorem 2.* By Lemma 8 we have

$$\begin{aligned} (F^*(\xi))(t) &= \bigvee_{x \in D(F)} (\langle \xi, x \rangle - F(x))(t) \\ &= \sup_{x \in D(F)} (\langle \xi(t), x \rangle - f(x, t)) \\ &= f^*(\xi(t), t) \quad (\text{a.e. } t \in \Omega), \end{aligned}$$

for every  $\xi \in D(F^*) \subset S(\Omega)^d$ . The latter statement can be obtained by analogy.

Combining Lemma 7 and Theorem 2, we obtain the following result.

**Theorem 3.** *A convex operator  $F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)$  satisfies*

$$F^{**}(x) = F(x)$$

*for every  $x \in D(F)$ , if and only if  $F$  has a normal representation.*



## REFERENCES

1. A. Brønsted, *An Introduction to Convex Polytopes.*, Springer-Verlag (1983).
2. N. Komuro, *On basic properties of convex functions and convex integrands.*, Hokkaido Math. J. **18** (1989), 1–30.
3. ———, *Convex operators and convex integrands*, Proc.Japan Acad. **65 A No.3** (1989), 77–80.
4. ———, *On the integrand representation of convex operators*, J. of Hokkaido Univ. of Educ. **46** (1996), 13–19.
5. S. S. Kutateladze, *Convex operators*, Russian Math. Survey **34** (1979).
6. R.T.Rockafellar, *Convex Analysis*, Princeton University Press (1970).
7. ———, *Integral functionals, Normal integrands, and Measurable selections*, Springer-Verlag Berlin (1976), 157–207.

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