EXISTENCE OF QUASIISOMETRIC MAPPINGS \\
AND ROYDEN COMPACTIFICATIONS  

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1. Introduction. Consider a $d$-dimensional $(d \geq 2)$ Riemannian manifold $D$ of class $C^\infty$ which is orientable and countable but not necessarily connected and given an exponent $1 < p < \infty$. The Royden $p$-algebra $M_p(D)$ of $D$ is defined by $M_p(D) := L^1_p(D) \cap L^\infty(D) \cap C(D)$, which is a commutative Banach algebra, i.e. the so-called normed ring, under pointwise addition and multiplication with $\|u; M_p(D)\| := \|u; L^\infty(D)\| + \|\nabla u; L^p(D)\|$ as norm, where $L^1_p(D)$ is the Dirichlet space, i.e. the space of locally integrable real valued functions $u$ on $D$ whose distributional gradients $\nabla u$ of $u$ belong to $L^p(D)$ considered with respect to the metric structure on $D$. The maximal ideal space $D'_p$ (cf. e.g. p.298 in [20]) of $M_p(D)$ is referred to as the Royden $p$-compactification of $D$, which is also characterized as the compact Hausdorff space containing $D$ as its open and dense subspace such that every function in $M_p(D)$ is continuously extended to $D'_p$ and $M_p(D)$ is uniformly dense in $C(D'_p)$ (cf. e.g. [17], [18], [11] and also p.154 in [14]).

Suppose that $D$ and $D'$ are $d$-dimensional $(d \geq 2)$ Riemannian manifolds of class $C^\infty$ which are orientable and countable but not necessarily connected. Moreover we always assume in this note that none of the components of $D$ and $D'$ is compact, which is however not an essential restriction and postulated only for the sake of simplicity. In 1982, the present author and H. Tanaka [13] (see also [10]) jointly showed that two conformal Royden compactifications $D'_d$ and $(D')'_d$ are homeomorphic if and only if there exists an almost quasiconformal mapping of $D$ onto $D'$. Here we say that a homeomorphism $f$ of $D$ onto $D'$ is an almost quasiconformal mapping of $D$ onto $D'$ if there exists a compact subset $E \subset D$ such that $f = f|D \setminus E$ is a quasiconformal mapping of $D \setminus E$ onto $D' \setminus f(E)$. There are many ways of defining quasiconformality but the following metric definition is convenient for applying to Riemannian manifolds (cf. e.g. p.113 in [19]): the homeomorphism $f$ of $D \setminus E$ onto $D' \setminus f(E)$ is quasiconformal, by definition, if

$$\sup_{x \in D \setminus E} \left( \limsup_{r \downarrow 0} \frac{\max_{\rho(x,y) = r} \rho'(f(x),f(y))}{\min_{\rho(x,y) = r} \rho'(f(x),f(y))} \right) < \infty,$$

where $\rho$ and $\rho'$ are geodesic distances on $D \setminus E$ and $D' \setminus f(E)$. It has been an open question for a long period since the above result was obtained as for what can be said about the

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counterpart of the above result for nonconformal case, i.e. if the exponent $d$ in the above result is replaced by $1 < p < d$. The purpose of this note is to settle this question by establishing the main theorem mentioned below.

To state our result we need to introduce a class of special kind of almost quasiconformal mappings. A homeomorphism $f$ of $D$ onto $D'$ is said to be an almost quasiisometric mapping of $D$ onto $D'$ if there exists a compact set $E \subset D$ such that $f = f|D \setminus E$ is a quasiisometric mapping of $D \setminus E$ onto $D' \setminus f(E)$. Here the homeomorphism $f$ of $D \setminus E$ onto $D' \setminus f(E)$ is quasiisometric, by definition, if there exists a constant $K \in [1, \infty)$ such that

$$\frac{1}{K} \rho(x, y) \leq \rho'(f(x), f(y)) \leq K \rho(x, y)$$

for every pair of points $x$ and $y$ in $D \setminus E$, where we always set $\rho(x, y) = \rho'(f(x), f(y)) = \infty$ if the component of $D \setminus E$ containing $x$ and that containing $y$ are different. From (3) it follows that

$$\frac{1}{K} r \leq \min_{\rho(x,y)=r} \rho'(f(x), f(y)) \leq \max_{\rho(x,y)=r} \rho'(f(x), f(y)) \leq Kr$$

for any fixed $x \in D$ and for any sufficiently small positive number $r > 0$, which implies that the left hand side term of (2) is dominated by $K^2$. Thus a quasiisometric mapping is automatically a quasiconformal mapping but obviously there exists a quasiconformal mapping which is not a quasiisometric mapping. Then our main result of this paper is stated as follows.

4. Main Theorem. When $1 < p < d$, Royden compactifications $D_p^*$ and $(D')_p^*$ are homeomorphic if and only if there exists an almost quasiisometric mapping of $D$ onto $D'$. More precisely, any almost quasiisometric mapping of $D$ onto $D'$ is uniquely extended to a homeomorphism of $D_p^*$ onto $(D')_p^*$; conversely, the restriction to $D$ of any homeomorphism of $D_p^*$ onto $(D')_p^*$ is an almost quasiisometric mapping of $D$ onto $D'$.

It may be interesting to compare the above topological result with the former relevant algebraic results obtained by the present author [8] and [9], Lewis [6], and Lelon-Ferrand [5] (cf. also Soderborg [15]): Royden algebras $M_D(D)$ and $M_D(D')$ are algebraically isomorphic if and only if there exists a quasiconformal mapping of $D$ onto $D'$; when $1 < p < d$, $M_p(D)$ and $M_p(D')$ are algebraically isomorphic if and only if there exists a quasiisometric mapping of $D$ onto $D'$. All these results including our present main theorem are shown to be invalid when $d < p < \infty$ by giving a counter example, which will be discussed elsewhere. Another important problem related to the above main result is the following: does the existence of an almost quasiisometric (almost quasiconformal, resp.) mapping of $D$ onto $D'$ imply that of a quasiisometric (quasiconformal, resp) mapping of $D$ onto $D'$? It is affirmative for the quasiconformal case if $D$ is the unit ball in the $d$-dimensional Euclidean space $\mathbf{R}^d$ (Gehring [2], see also Soderborg [16]); it is also affirmative again for the quasiconformal case if the dimensions of $D$ and $D'$ are 2. Except for these partial results though not easy to prove,
the problem is widely open.

5. Royden compactifications of Riemannian manifolds. By a Riemannian manifold $D$ of dimension $d \geq 2$ we always mean in this note an orientable and countable but not necessarily connected $C^\infty$ manifold $D$ of dimension $d$ with a metric tensor $(g_{ij})$ of class $C^\infty$. We also assume that any component of $D$ is not compact only for the sake of simplicity.

We say that $U$ or more precisely $(U, x)$ is a parametric domain on $D$ if the following two conditions are satisfied: firstly $U$ is a domain, i.e. a connected open set, in $D$; secondly $x$ is a $C^\infty$ diffeomorphism of $U$ onto a domain $x(U)$ in the Euclidean space $\mathbb{R}^d$ of dimension $d \geq 2$. The map $x = (x^1, \ldots, x^d)$ is referred to as a parameter on $U$. We often identify a generic point $P$ of $U$ with its parameter $x(P)$ and denote them by the same letter $x$, for example. In other words we view $U$ to be embedded in $\mathbb{R}^d$ by identifying $U$ with $x(U)$ so that $U$ itself may be considered as a Riemannian manifold $(U, g_{ij})$ with metric tensor $(g_{ij})$ restricted on $U$ and at the same time as an Euclidean subdomain $(U, \delta_{ij})$ with the natural metric tensor $(\delta_{ij})$, $\delta_{ij}$ being the Kronecker delta.

Take a parametric domain $(U, x)$ on $D$. The metric tensor $(g_{ij})$ on $D$ gives rise to a $d \times d$ matrix $(g_{ij}(x))$ of functions $g_{ij}(x)$ on $U$. We say that $(U, x)$ is a $\lambda$-domain with $\lambda \in [1, \infty)$ if the following matrix inequalities hold:

\begin{equation}
\frac{1}{\lambda}(\delta_{ij}) \leq (g_{ij}(x)) \leq \lambda(\delta_{ij})
\end{equation}

for every $x \in U$. It is important that any point of $D$ has a $\lambda$-domain as its neighborhood for any $\lambda \in (1, \infty)$. This comes from the fact that there exists a parametric ball $(U, x)$ at any point $P \in D$ (i.e. a parametric domain $(U, x)$ such that $x(P) = 0$ and $x(U)$ is a ball in $\mathbb{R}^d$ centered at the origin $0$) such that $(g_{ij}(x))$ satisfies $g_{ij}(0) = \delta_{ij}$.

The metric tensor $(g_{ij})$ on $D$ defines the line element $ds$ on $D$ by $ds^2 = g_{ij}(x)dx^i dx^j$ in each parametric domain $(U, x = (x^1, \ldots, x^d))$. Here and hereafter we follow the Einstein convention: whenever an index $i$ appears both in the upper and lower positions, it is understood that summation for $i = 1, \ldots, d$ is carried out. The length of a rectifiable curve $\gamma$ on $D$ is given by $\int_{\gamma} ds$. The geodesic distance $\rho(x, y)$ between two points $x$ and $y$ in $D$ is given by

$$
\rho(x, y) = \rho_D(x, y) = \inf_{\gamma} \int_{\gamma} ds,
$$

where the infimum is taken with respect to rectifiable curves $\gamma$ connecting $x$ and $y$. Needless to say, if there is no such curve $\gamma$, i.e. if $x$ and $y$ are in the different components of $D$, then, as the infimum of empty set, we understand that $\rho(x, y) = \infty$. When $(U, x)$ is a parametric domain and considered as the Riemannian manifold $(U, \delta_{ij})$, then $\rho_U(x, y)$ can also be given by

$$
\rho(x, y) = \rho_U(x, y) = \inf_{i=0}^{n} |x_i - x_{i-1}|,
$$

for any $x = (x_0, \ldots, x_n)$ and $y = (y_0, \ldots, y_n)$ in $U$.
where the infimum is taken with respect to every polygonal line $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ such that every line segment $[x_{i-1}, x_i] = \{(1 - t)x_{i-1} + tx_i : 0 \leq t \leq 1\} \subset U$ for each $i = 1, \ldots, n$.

We write $(g^{ij}) := (g_{ij})^{-1}$ and $g := \det(g_{ij})$. We denote by $dV$ the volume element on $D$ so that
\[
dV(x) = \sqrt{g(x)}dx^1 \wedge \cdots \wedge dx^d
\]
in each parametric domain $(U, x = (x^1, \ldots, x^d))$. On $(U, \delta_{ij})$ we also have the volume element (Lebesgue measure) $dx = dx^1 \cdots dx^d$. Sometimes we use $dx$ to mean $(dx^1, \ldots, dx^d)$ but there will be no confusion by context. The Riemannian volume element $dV(x)$ and the Euclidean (Lebesgue) volume element $dx$ are mutually absolutely continuous and the Radon-Nikodym densities $dV(x)/dx = \sqrt{g(x)}$ and $dx/dV(x) = 1/\sqrt{g(x)}$ are locally bounded on $U$. Thus a.e.$dV$ and a.e.$dx$ are identical and we can loosely use a.e. without referring to $dV$ or $dx$.

For each $x \in D$, the tangent space to $D$ at $x$ will be denoted by $T_x D$. We denote by $\langle h, k \rangle$ the inner product of two tangent vectors $h$ and $k$ in $T_x D$ and by $|h|$ the length of $h \in T_x D$ so that if $(h_1, \ldots, h_d)$ and $(k_1, \ldots, k_d)$ are covariant components of $h$ and $k$, then
\[
\langle h, k \rangle = g^{ij}h_i k_j \quad \text{and} \quad |h| = \langle h, h \rangle^{1/2} = (g^{ij}h_i h_j)^{1/2}.
\]

Since we may consider two metric tensors $(g_{ij})$ and $(\delta_{ij})$ on a parametric domain $(U, x)$, we occasionally write $\langle h, k \rangle_{g_{ij}}$ or $\langle h, k \rangle_{\delta_{ij}}$ and similarly $|h|_{g_{ij}}$ or $|h|_{\delta_{ij}}$ to make clear whether they are considered on $(U, g_{ij})$ or on $(U, \delta_{ij})$.

Let $G$ be an open subset of $D$. In this note we use the notation $L^p(G)$ ($1 \leq p \leq \infty$) in two ways. The first is the standard use: $L^p(G) = L^p(G; g_{ij})$ is the Banach space of measurable functions $u$ on $G$ with the finite norm $\|u; L^p(G)\|$ given by
\[
\|u; L^p(G)\| := \left( \int_G |u|^p dV \right)^{1/p} \quad (1 \leq p < \infty)
\]
and $\|u; L^\infty(G)\|$ is the essential supremum of $|u|$ on $G$. The second use: for a measurable vector field $X$ on $G$ we write $X \in L^p(G) = L^p(G; g_{ij})$ if $|X| = |X|_{g_{ij}} \in L^p(G)$ in the first sense and we set
\[
\|X; L^p(G)\| := \||X|; L^p(G)\|.
\]
The Dirichlet space $L^{1,p}(G) = L^{1,p}(G; g_{ij})$ ($1 \leq p \leq \infty$) is the class of functions $u \in L^1_{\text{loc}}(G)$ with the distributional gradients $\nabla u \in L^p(G)$, where the distributional gradient $\nabla u$ is determined by the relation
\[
\int_G (\nabla u, \Psi) dV = -\int_G u \text{div} \Psi dV
\]
for every $C^\infty$ vector field $\Psi$ on $G$ with compact support in $G$. In the parametric domain $(U, x)$ in $G$ we have $\nabla u = (\partial u/\partial x^1, \ldots, \partial u/\partial x^d)$. If $\Psi = (\psi_1, \ldots, \psi_d)$ in $U$, then
\[
\text{div} \Psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} g^{ij} \psi_j).
\]
The Sobolev space \( W^{1,p}(G) = W^{1,p}(G, g_{ij}) \) \((1 \leq p \leq \infty)\) is the Banach space \( L^{1,p}(G) \cap L^{p}(G) \) equipped with the norm
\[
\|u; W^{1,p}(G)\| := \|u; L^{p}(G)\| + \|\nabla u; L^{p}(G)\|.
\]

Given a Riemannian manifold \( D \) of dimension \( d \geq 2 \) and given an exponent \( 1 < p < \infty \), the Royden \( p \)-algebra \( M_{p}(D) \) is the Banach algebra \( L^{1,p}(D) \cap L^{\infty}(D) \cap C(D) \) equipped with the norm
\[
\|u; M_{p}(D)\| := \|u; L^{\infty}(D)\| + \|\nabla u; L^{p}(D)\|.
\]

By the standard mollifier method we can show that the subalgebra \( M_{p}(D) \cap C^{\infty}(D) \) is dense in \( M_{p}(D) \) with respect to the norm in (7). Hence \( M_{p}(D) \) may also be defined as the completion of \( \{u \in C^{\infty}(D) : \|u; M_{p}(D)\| < \infty\} \) without appealing to the Dirichlet space. It is important that \( M_{p}(D) \) is closed under lattice operations \( \cup \) and \( \cap \) given by \((u \cup v)(x) = \max(u(x), v(x))\) and \((u \cap v)(x) = \min(u(x), v(x))\) (cf. e.g. p.21 in [4]). The maximal ideal space \( D_{p}^{\ast} \) of \( M_{p}(D) \) is referred to as the Royden \( p \)-compactification, which can also be characterized as the compact Hausdorff space containing \( D \) as its open and dense subspace such that every function \( u \in M_{p}(D) \) is continuously extended to \( D_{p}^{\ast} \) and \( M_{p}(D) \), viewed as a subspace of \( C(D_{p}^{\ast}) \) by this continuous extension, is dense in \( C(D_{p}^{\ast}) \) with respect to its supremum norm.

8. Capacities of rings. A ring \( R \) in a Riemannian manifold \( D \) is a subset \( R \) of \( D \) with the following properties: \( R \) is a subdomain of \( D \) so that \( R \) is contained in a unique component \( D_{R} \) of \( D \); \( D_{R} \setminus R \) consists of exactly two components one of which, denoted by \( C_{1} \), is compact and the other of which, denoted by \( C_{0} \), is noncompact. The set \( C_{1} \) will be referred to as the inner part of \( R^{c} := D \setminus R \) and the set \( D \setminus (R \cup C_{0}) \) as the outer part of \( R^{c} \). We denote by \( W(R) \) the class of functions \( u \in W_{1,1}^{1}(R) \cap C(D) \) such that \( u = 1 \) on the inner part of \( R^{c} \) and \( u = 0 \) on the outer part of \( R^{c} \) which includes \( C_{0} \). The \( p \)-capacity \( \text{cap}_{p}R \) \((1 \leq p \leq \infty)\) of the ring \( R \subset D \) is given by
\[
\text{cap}_{p}R := \inf_{u \in W(R)} \|\nabla u; L^{p}(R)\|^{p}
\]
for \( 1 \leq p < \infty \) and \( \text{cap}_{\infty}R := \inf_{u \in W(R)} \|\nabla u; L^{\infty}(R)\| \). Note that \( \text{cap}_{p}R \) does not depend upon which Riemannian manifold \( D \) the ring \( R \) is embedded as far as the metric structure on \( R \) is unaltered. The following inequality will be essentially made use of (cf. e.g. p.32 in [4]): if \( 1 < p < \infty \) and if \( R \) is a ring in \( D \) and \( R_{k} \) \((1 \leq k \leq n)\) are disjoint rings contained in \( R \) each of which separates the boundary components of \( R \), then
\[
(\text{cap}_{p}R)^{\frac{1}{p}} \geq \sum_{k=1}^{n} (\text{cap}_{p}R_{k})^{\frac{1}{p}}.
\]
Suppose that a ring $R$ is contained in a parametric domain $(U, x)$ on $D$ for which two metric structures $(g_{ij})$ and $(\delta_{ij})$ can be considered. If the need occurs to indicate that $\text{cap}_p R$ is considered on $(U, \delta_{ij})$, then we write

$$\text{cap}_p R = \text{cap}_p(R, \delta_{ij}) = \inf_{u \in W(R)} \int_R |\nabla u(x)|_{\delta_{ij}}^p dx;$$

if $\text{cap}_p R$ is considered on $(U, g_{ij})$, then we write

$$\text{cap}_p R = \text{cap}_p(R, g_{ij}) = \inf_{u \in W(R)} \int_R |\nabla u|^p_{g_{ij}} dV$$

for $1 \leq p < \infty$. Similar considerations are applied to $\text{cap}_\infty(R, g_{ij})$ and $\text{cap}_\infty(R, \delta_{ij})$. If moreover $U$ is a $\lambda$-domain for any $\lambda \in [1, \infty)$, then (6) implies that

$$\frac{1}{\lambda^{d+p/2}} \text{cap}_p(R, \delta_{ij}) \leq \text{cap}_p(R, g_{ij}) \leq \lambda^{d+p/2} \text{cap}_p(R, \delta_{ij}).$$

In the case $p = \infty$, the inequality corresponding to the above takes the following form:

$$\lambda^{-1/2} \text{cap}_\infty(R, \delta_{ij}) \leq \text{cap}_\infty(R, g_{ij}) \leq \lambda^{1/2} \text{cap}_\infty(R, \delta_{ij}),$$

which however will not be used in this note.

We fix a parametric domain $(U, x)$ in $D$. It is possible that the parametric domain is the $d$-dimensional Euclidean space $\mathbb{R}^d$ itself. A ring $R$ contained in $U$ is said to be a spherical ring in $(U, x)$ if

$$R = \{x \in U : a < |x - P| < b\},$$

where $P \in U$ and $a$ and $b$ are positive numbers with $0 < a < b < \inf_U |x - P|$. At this point we must be careful: in the case where the above $R$ happens to be included in another parametric domain $(V, y)$ of $D$, $R$ may not be a spherical ring in $(V, y)$ even if $R$ is a spherical ring in $(U, x)$. Namely, the notion of spherical rings cannot be introduced to the general Riemannian manifold $D$ and is strictly attached to the parametric domain in question. Let $R$ be a spherical ring in a parametric domain $(U, x)$ with the above expression (12). Then we have (cf. e.g. p.35 in [4])

$$\text{cap}_p R = \text{cap}_p(R, \delta_{ij}) = \begin{cases} 
\omega_d \left( \frac{b^q - a^q}{q} \right)^{1-p} & (1 < p < \infty, p \neq d), \\
\omega_d \left( \log \frac{b}{a} \right)^{1-d} & (p = d), 
\end{cases}$$

where we have set $q = (p - d)/(p - 1)$ and $\omega_d$ is the surface area of the Euclidean unit sphere $S^{d-1}$. In passing we state that $\text{cap}_1(R, \delta_{ij}) = \omega_d a^{d-1}$ and $\text{cap}_\infty(R, \delta_{ij}) = 1/(b - a)$, which are also not used in this note.
Another important ring in $\mathbb{R}^d$ which we use later is a Teichmüller ring $R_T$ defined by $R_T = \mathbb{R}^d \setminus \{te_1 : t \in [-1, 0] \cup [1, \infty)\}$, where $e_1$ is the unit vector $(1, 0, \cdots, 0)$ in $\mathbb{R}^d$. We set

(14) \hspace{1cm} t_d := \text{cap}_d(R_T, \delta_i).

Finally in this section we state a separation lemma on the topology of the Royden compactification. Let $(R_n)_{n \geq 1}$ be a sequence of rings $R_n$ in $D$ ($n = 1, 2, \cdots$) with the following properties: $(R_n \cup C_{n1}) \cap (R_m \cup C_{m1}) = \emptyset$ for $n \neq m$, where $C_{n1}$ is the inner part of $(R_n)^c = D \setminus R_n$; $(R_n)_{n \geq 1}$ does not accumulate in $D$, i.e. $\{n : E \cap (\overline{R_n} \cup C_{n1}) \neq \emptyset\}$ is a finite set for any compact set $E$ in $D$. Such a sequence $(R_n)_{n \geq 1}$ will be called an admissible sequence, which defines two disjoint closed sets $X$ and $Y$ in $D$ as follows:

\[ X := \bigcup_{n=1}^{\infty} C_{n1} \quad \text{and} \quad Y := \bigcap_{n=1}^{\infty} (D \setminus (R_n \cup C_{n1})). \]

We denote by $\text{cl}(X; D^*_p)$ the closure of $X$ in $D^*_p$. Although $X \cap Y = \emptyset$ in $D$, $\text{cl}(X; D^*_p)$ and $\text{cl}(Y; D^*_p)$ may intersect on the Royden $p$-boundary

\[ \Gamma_p(D) := D^*_p \setminus D. \]

Concerning to this we have the following result.

15. LEMMA. The set $\text{cl}(\bigcup_{n=1}^{\infty} R_n; D^*_p)$ for an admissible sequence $(R_n)_{n \geq 1}$ in $D$ separates $\text{cl}(X; D^*_p)$ and $\text{cl}(Y; D^*_p)$ in $D^*_p$ in the sense that

(16) \hspace{1cm} (\text{cl}(X; D^*_p)) \cap (\text{cl}(Y; D^*_p)) = \emptyset

if and only if

(17) \hspace{1cm} \sum_{n=1}^{\infty} \text{cap}_p R_n < \infty.

PROOF: First we show that (16) implies (17). By (16) the Urysohn theorem assures the existence of a function $u \in C(D^*_p)$ such that $u = 3$ on $\text{cl}(X; D^*_p)$ and $u = -2$ on $\text{cl}(Y; D^*_p)$. Since $M_p(D)$ is dense in $C(D^*_p)$, there is a function $v \in M_p(D)$ such that $v > 2$ on $X$ and $v < -1$ on $Y$. Finally let $w = ((v \cap 1) \cup 0) \in M_p(D)$, which satisfies $w|X = 1$, $w|Y = 0$ and $0 \leq w \leq 1$ on $D$. Set $w_n = w$ on $R_n \cup C_{n1}$ and $w_n = 0$ on $D \setminus (R_n \cup C_{n1})$ for $n = 1, 2, \cdots$. Clearly $w_n \in W(R_n)$ so that $\text{cap}_p R_n \leq \|\nabla w_n; L^p(R_n)\|_p$ ($n = 1, 2, \cdots$) and $w = \sum_{n=1}^{\infty} w_n$. Since the supports of $w_n$ in $D$ ($n = 1, 2, \cdots$) are mutually disjoint, we see that

\[ \sum_{n=1}^{\infty} \text{cap}_p R_n \leq \sum_{n=1}^{\infty} \|\nabla w_n; L^p(R_n)\|_p = \|\nabla w; L^p(D)\|_p \leq \|w; M_p(D)\|_p < \infty, \]

i.e. (17) has been deduced.
Conversely, suppose that (17) is the case. We wish to derive (16) from (17). Choose a function \( w_n \in W(R_n) \) such that \( \| \nabla w_n; L^p(R_n) \|^p < 2 \text{cap}_p R_n \) for each \( n = 1, 2, \cdots \). We may suppose that \( 0 \leq w_n \leq 1 \) on \( D \) by replacing \( w_n \) with \( (w_n \cap 1) \cup 0 \) if necessary (see e.g. p.20 in [4]). Clearly \( w := \sum_{n=1}^{\infty} w_n \in M_p(D) \) since \( \| w; L^\infty(D) \| = 1 \) and
\[
\| \nabla w; L^p(D) \|^p = \sum_{n=1}^{\infty} \| \nabla w_n; L^p(D_n) \|^p \leq 2 \sum_{n=1}^{\infty} \text{cap}_p R_n < \infty.
\]
Observe that \( w = 1 \) on \( X \) and \( w = 0 \) on \( Y \). Hence, by the continuity of \( w \) on \( D^*_p \), we see that \( w = 1 \) on \( \text{cl}(X; D^*_p) \) and \( w = 0 \) on \( \text{cl}(Y; D^*_p) \), which yields (16).

As a consequence of the separation lemma above we can characterize points in the Royden p-boundary \( \Gamma_p(D) = D^*_p \setminus D \) among points in \( D^*_p \) in terms of their being not \( G_\delta \) for \( 1 \leq p \leq d \). This is no longer true for \( d < p \leq \infty \). Recall that a point \( \zeta \in D^*_p \) is said to be \( G_\delta \) if there exists a countable sequence \( (\Omega_i)_{i \geq 1} \) of open neighborhoods \( \Omega_i \) of \( \zeta \) such that \( \cap_{i \geq 1} \Omega_i = \{ \zeta \} \).

18. **Corollary to Lemma 15.** A point \( \zeta \) in \( D^*_p \) (1 \leq p \leq d) belongs to \( D \) if and only if \( \zeta \) is \( G_\delta \).

**Proof:** We only have to show that \( \zeta \in \Gamma_p(D) = D^*_p \setminus D \) is not \( G_\delta \). Contrariwise suppose \( \zeta \) is \( G_\delta \) so that there exists a sequence \( (\Omega_i)_{i \geq 1} \) of open neighborhoods of \( \zeta \) such that \( \Omega_i \supset \text{cl}(\Omega_{i+1}; D^*_p) \) \( (i = 1, 2, \cdots) \) and \( \cap_{i \geq 1} \Omega_i = \{ \zeta \} \). Since \( D \) is dense in \( D^*_p \), \( H_i := D \cap (\Omega_i \setminus \text{cl}(\Omega_{i+1}; D^*_p)) \) is a nonempty open subset of \( D \) for each \( i \). Hence we can find a sequence \( (P_n)_{n \geq 1} \) of points \( P_n \in H_n \) \( (n = 1, 2, \cdots) \) and a sequence \( ((U_n, x_n))_{n \geq 1} \) of 2-domains \( (U_n, x_n) \) contained in \( H_n \) \( (n = 1, 2, \cdots) \) such that \( U_n = \{ x_n : |x_n - P_n| < r_n \} \) \( (r_n > 0) \) \( (n = 1, 2, \cdots) \). Let \( R_n := \{ x_n : a_n < |x_n - P_n| < b_n \} \) \( (0 < a_n < b_n := r_n/2) \) be a spherical ring in \( (U_n, x_n) \). Clearly \( (R_n)_{n \geq 1} \) is an admissible sequence. Since \( \text{cap}_p(R_n, \delta_{ij}) = \omega_d ((1 - (a_n/b_n))^{p-1 - \delta} a_n^{d-\delta} b_n^{\delta}) \) by (13) for \( 1 < p < d \), \( \text{cap}_p(R_n, \delta_{ij}) = \omega_d/(b_n/a_n)^{d-1} \) and \( \text{cap}(R_n, \delta_{ij}) = \omega_d b_n^{d-1} \), we can see that \( \text{cap}_p(R_n, \delta_{ij}) < 2^{-n} \) by choosing \( a_n \in (0, r_n/2) \) enough small so that \( \text{cap}_p R = \text{cap}_p(R, g_{ij}) \leq 2^{(d+p)/2} \text{cap}_p(R, \delta_{ij}) < 2^{(d+p)/2} \) \( (n = 1, 2, \cdots) \) by (11). Hence (17) is satisfied but (16) is invalid because the intersection on the left hand side of (16) contains \( \zeta \) due to the fact that \( R_n \subset H_n \) \( (n = 1, 2, \cdots) \). This is clearly a contradiction to Lemma 15.

19. **Analytic properties of quasiisometric mappings.** A quasiisometric (quasi-conformal, resp.) mapping \( f \) of a Riemannian manifold \( D \) onto another \( D' \) is, as defined in §1 (Introduction), a homeomorphism \( f \) of \( D \) onto \( D' \) such that \( K^{-1} \rho(x, y) \leq \rho(f(x), f(y)) \leq K \rho(x, y) \) for every pair of points \( x \) and \( y \) in \( D \) for some fixed \( K \in [1, \infty) \) \((\sup_{x \in D} \lim \sup_{r \downarrow 0} ((\max_{\rho(x,y)>r}) \rho'(f(x), f(y)))/\min_{\rho(x,y)>r} \rho'(f(x), f(y))) < \infty \), resp.), where \( \rho \) and \( \rho' \) are geodesic distances on \( D \) and \( D' \), respectively. In this case we also say that \( f \) is \( K \)-quasiisometric referring to \( K \). For simplicity, quasiisometric (quasiconformal, resp.) mappings will occasionally be abbreviated as qi (qc, resp.). Consider a \( K \)-qi \( f \) of a \( d \)-
dimensional Riemannian manifold $D$ equipped with the metric tensor $(g_{ij})$ onto another $d$-dimensional Riemannian manifold $D'$ equipped with the metric tensor $(g'_{ij})$. Fix an arbitrary $\lambda \in (0, \infty)$ and choose any $\lambda$-domain $(U, x)$ in $D$ and any $\lambda$-domain $(U', x')$ in $D'$ such that $f(U) = U'$. The mapping $f : (U, \delta_{ij}) \to (U', \delta_{ij})$ has the representation

(20) $x' = f(x) = (f^1(x), \cdots, f^d(x))$

on $U$ in terms of the parameters $x$ and $x'$. As the composite mapping of $id. : (U, \delta_{ij}) \to (U, g_{ij}), f : (U, g_{ij}) \to (U', g'_{ij})$, and $id. : (U', g'_{ij}) \to (U', \delta_{ij})$, we see that the mapping $f : (U, \delta_{ij}) \to (U', \delta_{ij})$ is $\lambda K$-qi since $id. : (U, \delta_{ij}) \to (U, g_{ij})$ and $id. : (U', g'_{ij}) \to (U', \delta_{ij})$ are $\sqrt{\lambda}$-qi as the consequence of $\lambda^{-1}|dx|^2 \leq ds^2 \leq \lambda |dx|^2$, where $dx = (dx^1, \cdots, dx^d)$, $|dx|^2 = \delta_{ij}dx^i dx^j$, and $ds^2 = g_{ij}(x)dx^i dx^j$, which is deduced from $\lambda^{-1}(\delta_{ij}) \leq (g_{ij}) \leq \lambda(\delta_{ij})$. Hence we see that

(21) $\frac{1}{\lambda K}|x - y| \leq |f(x) - f(y)| \leq \lambda K|x - y|$

whenever the line segment $[x, y] := \{(1-t)x + ty : t \in [0,1]\} \subset U$ and $[f(x), f(y)] \subset U'$. In particular (21) implies that

(22) $\limsup_{h \to 0} \frac{|f(x + h) - f(x)|}{|h|} \leq \lambda K < \infty$

for every $x \in U$ and

(23) $\liminf_{h \to 0} \frac{|f(x + h) - f(x)|}{|h|} \geq \frac{1}{\lambda K} > 0$.

As an important consequence of (22), the Rademacher-Stepanoff theorem (cf. e.g. p.218 in [1]) assures that $f(x)$ is differentiable at a.e. $x \in U$, i.e.

(24) $f(x + h) - f(x) = f'(x)h + \varepsilon(x, h)|h| \quad (\lim_{h \to 0} \varepsilon(x, h) = 0)$

for a.e. $x \in U$, where $f'(x)$ is the $d \times d$ matrix $(\partial f^i / \partial x^j)$. Fix an arbitrary vector $h$ with $|h| = 1$. Then for any positive number $t > 0$ we have, by replacing $h$ in(24) with $th$,

$$|f'(x)h| - |\varepsilon(x, th)| \leq \frac{|f(x + th) - f(x)|}{|th|}$$

and on letting $t \downarrow 0$ we obtain by (22) that $|f'(x)h| \leq \lambda K$. Therefore

(25) $|f'(x)| := \sup_{|h| = 1} |f'(x)h| \leq \lambda K$

for a.e. $x \in U$. Similarly we have

$$|f'(x)h| + |\varepsilon(x, th)| \geq \frac{|f(x + th) - f(x)|}{|th|}$$
and hence by (23) we deduce $|f'(x)h| \geq 1/\lambda K$. Hence

$$l(f'(x)) := \inf_{|h|=1} |f'(x)h| \geq \frac{1}{\lambda K}. \tag{26}$$

From (25) it follows that $|\partial f'(x)/\partial x^i| \leq |f'(x)| \leq \lambda K$ for a.e. $x \in U$ ($i, j = 1, \cdots, d$) and thus $|\nabla f| = (\sum_{i=1}^{d} |\nabla f_i|^2)^{1/2} \in L^\infty(U)$. By (21), $f(x)$ is ACL (absolutely continuous on almost all straight lines which are parallel to coordinate axes). That $f(x)$ is ACL and $\nabla f \in L^\infty(U)$ is necessary and sufficient for $f$ to belong to $L^{1,\infty}(U)$ (cf. e.g. pp.8-9 in [7]) so that, by the continuity of $f$, we have

$$f \in W^{1,\infty}_{loc}(D). \tag{27}$$

By (25) and (26) we have the matrix inequality

$$l(f'(x))^2(\delta_{ij}) \leq f'(x)^*f'(x) \leq |f'(x)|^2(\delta_{ij})$$

for a.e. $x \in U$, where $f'(x)^*$ is the transposed matrix of $f'(x)$. Let $\lambda_1(x) \geq \cdots \geq \lambda_d(x)$ be the square roots of the proper values of the symmetric positive matrix $f'(x)^*f'(x)$. Then

$$\frac{1}{\lambda K} \leq l(f'(x)) = \lambda_d(x) \leq \cdots \leq \lambda_1(x) = |f'(x)| \leq \lambda K.$$

Observe that $\prod_{i=1}^{d} \lambda_i(x)^2 = \det(f'(x)^*f'(x)) = (\det f'(x))^2$ is the square of the Jacobian $J_f(x)$ of $f$ at $x$. Hence, by $\lambda K \lambda_i \geq 1$ ($i = 2, 3, \cdots, d$), we see that

$$|f'(x)|^p = \lambda_1(x)^p \leq \lambda_1(x)(\lambda K)^{p-1} \leq \lambda_1(x)(\lambda K)^{p-1} \prod_{i=2}^{d} (\lambda K \lambda_i(x))$$

$$= (\lambda K)^{d+p-2} \prod_{i=1}^{d} \lambda_i(x) = (\lambda K)^{d+p-2}|J_f(x)|,$$

i.e. we have deduced that

$$|f'(x)|^p \leq (\lambda K)^{d+p-2}|J_f(x)| \tag{28}$$

for a.e. $x \in U$. This is used to prove the following result.

29. **Proposition.** The pull-back $v = u \circ f$ of any $u$ in $M_p(D')$ by a $K$-qi $f$ of $D$ onto $D'$ belongs to $M_p(D)$ and satisfies the inequality

$$\int_{D} |\nabla v(x)|_{g_{ij}}^p \sqrt{g(x)} dx \leq K^{d+p-2} \int_{D'} |\nabla u(x')|_{g'_{ij}}^p \sqrt{g'(x')} dx' \tag{30}$$

and in particular

$$\|v; M_p(D)\| \leq K^{(d+p-2)/p}\|u; M_p(D')\|. \tag{31}$$
PROOF: The inequality (30) is nothing but \( \| \nabla v; L^p(D) \| \leq K^{(d+p-2)/p} \| \nabla v; L^p(D') \| \). This with \( \| v; L^\infty(D) \| = \| u; L^\infty(D') \| \) implies (31). Suppose that Proposition 29 is true if \( u \in M_p(D') \cap C^\infty(D') \). Since \( M_p(D') \cap C^\infty(D') \) is dense in \( M_p(D') \), for an arbitrary \( u \in M_p(D') \), there exists a sequence \( (u_k)_{k \geq 1} \) in \( M_p(D') \cap C^\infty(D') \) such that \( \| u - u_k; M_p(D') \| \to 0 \) \( (k \to \infty) \). In particular \( \| u_k - u_{k'}; M_p(D') \| \to 0 \) \( (k, k' \to \infty) \). By our assumption, \( u_k := u_k \circ f \in M_p(D) \) \( (k = 1, 2, \cdots) \). By (31), the inequalities \( \| v_k - v_{k'}; M_p(D) \| \leq K^{(d+p-2)/p} \| u_k - u_{k'}; M_p(D') \| \) assure that \( \| v_k - v_{k'}; M_p(D) \| \to 0 \) \( (k, k' \to \infty) \). By the completeness of \( M_p(D) \), since \( \| v - v_k; L^\infty(D) \| \to 0 \) \( (k \to \infty) \), we see that \( v \in M_p(D) \).

By the validity of (30) \( \lambda \in (1, \infty) \) for \( u_k \), we see that (30) is valid for \( v \). For this reason we can assume \( u \in M_p(D') \cap C^\infty(D') \) to prove Proposition 29.

It is clear by (25) that \( v = u \circ f \in W^{1,\infty}_{1,\text{loc}} \cap L^\infty(D) \cap C(D) \) if \( u \in M_p(D') \cap C^\infty(D') \). Hence we only have to prove (30) to deduce (30) \( \lambda \). Let \( D = \bigcup_{k=1}^\infty E_k \) be a union of disjoint Borel sets \( E_k \) in \( D \) such that each \( E_k \) is contained in a \( \lambda \)-domain \( U_k \) in \( D \) and \( E_k = f(E_k) \) in a \( \lambda \)-domain \( U_k' = f(U_k) \) in \( D' \) for \( k = 1, 2, \cdots \). Fix a \( k \) and consider the \( \lambda K \)-qi \( f \) of \( (U_k, \delta_{ij}) \) onto \( (U_k', \delta_{ij}) \) with the representation (20) on \( U_k \) in terms of the parameter \( x \) in \( U_k \) and \( x' \) in \( U_k' \). By the chain rule we have

\[
(32) \quad \nabla v(x) = f'(x)^* \nabla u(f(x))
\]

for a.e. \( x \in U_k \). Since \( |f'(x)|^* = |f'(x)| \), (28) and (32) yield

\[
|\nabla v(x)|^p \leq (\lambda K)^{d+p-2}|\nabla u(f(x))|^p|J_f(x)|
\]

for a.e. \( x \in U_k \). In view of (22), the formula of the change of variables in integrations is valid for \( x' = f(x) \):

\[
\int_{E_k} |\nabla u(f(x))|^p|J_f(x)|dx = \int_{E_k'} |\nabla u(x')|^pdx'.
\]

From the above two displayed relations we deduce

\[
\int_{E_k} |\nabla v(x)|^pdx \leq (\lambda K)^{d+p-2} \int_{E_k'} |\nabla u(x')|^pdx'.
\]

Observe that \( |\nabla v|_{g_{ij}}^p \leq \lambda^{p/2} |\nabla u|^p \) and \( \sqrt{g} \leq \lambda^{d/2} \), and similarly, that \( |\nabla u|^p \leq \lambda^{p/2} |\nabla u|_{g_{ij}}^p \) and \( 1 \leq \lambda^{d/2} \sqrt{g} \). The above displayed inequality then implies that

\[
\int_{E_k} |\nabla v(x)|_{g_{ij}}^p \sqrt{g(x)}dx \leq \lambda^{2(d+p-1)} K^{d+p-2} \int_{E_k} |\nabla u(x')|^p \sqrt{g'(x')}dx'.
\]

On adding these inequalities for \( k = 1, 2, \cdots \) we obtain (30) with \( K^{d+p-2} \) replaced by \( \lambda^{2(p+d-1)} K^{d+p-2} \). Since \( \lambda \in (1, \infty) \) is arbitrary, we deduce (30) itself by letting \( \lambda \downarrow 1 \). \( \square \)

33. Distortion of rings and their capacities. Throughout this section we fix two nonempty open sets \( V \) and \( V' \) in \( \mathbb{R}^d \) (or, what amounts to the same, two parametric domains
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Lemma 37 can be proven by suitably modifying the original Gehring proof ([3]) of his theorem. A complete proof of Lemma 37 can be found in [12].

If we assume that $f$ is $K_1$-qi, then $f, f^{-1} \in Lip(K_1)$, which is the conclusion of Lemma 37, follows immediately. We now prove the converse of this so that $f, f^{-1} \in Lip(K)$ can be used for the definition of $K$-qi in the case of mappings between space open sets.

39. LEMMA. If $f, f^{-1} \in Lip(K)$, then $f$ is a $K$-qi of $V$ onto $V'$.

PROOF: We define positive numbers $s(r) > 0$ for sufficiently small positive numbers $r > 0$ by $\min_{|x-P|=r} |f(x) - f(P)| =: s(r)$ for an arbitrarily fixed $P \in V$. On setting $P' := f(P)$ we see that $\max_{|x'-P'|=s(r)} |f^{-1}(x') - f^{-1}(P')| = r$. Observe that $s(r) \downarrow 0$ as $r \downarrow 0$. Hence, by $f^{-1} \in Lip(K) = Lip(K; V', V)$, we see that

$$\lim_{r \downarrow 0} \sup_{s \downarrow 0} \frac{r}{s(r)} \leq K.$$ 

Therefore we infer that

$$\lim_{r \downarrow 0} \sup_{s \downarrow 0} \frac{\max_{|x-P|=r} |f(x) - f(P)|}{\min_{|x-P|=r} |f(x) - f(P)|} = \lim_{r \downarrow 0} \left( \frac{\max_{|x-P|=r} |f(x) - f(P)|}{r} \cdot \frac{r}{s(r)} \right) \leq K^2,$$

which concludes that $f$ is a qc of $V$ onto $V'$ by the metric definition (2) of quasiconformality. This assures that $f$ is differentiable a.e. on $V$ and $f \in W_{loc}^{1,d}(V)$ (cf. e.g. pp.109-111 in [19]). The latter in particular implies that $f$ is ACL in an arbitrarily given direction $l$: $f$ is absolutely continuous on almost all straight lines which are parallel to $l$. Suppose that $f$ is differentiable at $x \in V$, i.e.

$$f(x + h) - f(x) = f'(x)h + \epsilon(x, h)|h| \quad (\lim_{h \to 0} \epsilon(x, h) = 0).$$

For any $|h| = 1$ and any small $t > 0$, we have

$$|f'(x)h| \leq \frac{|f(x + th) - f(x)|}{|th|} + |\epsilon(x, th)| \leq \frac{\max_{|y-x|=t} |f(y) - f(x)|}{t} + |\epsilon(x, th)|.$$

On letting $t \downarrow 0$ we deduce $|f'(x)h| \leq K$ since $f \in Lip(K)$. We can thus conclude that

$$(40) \quad |f'(x)| = \sup_{|h|=1} |f'(x)h| \leq K.$$
for a.e. $x \in U$. We now maintain that

\begin{equation}
|f(x) - f(y)| \leq K|x - y|
\end{equation}

for any line segment $[x, y] = \{(1-t)x + ty : t \in [0, 1]\} \subset V$. Since $f$ is ACL in the direction of $[x, y]$, we see that $f$ is absolutely continuous in $V$ on almost all straight lines $L$ parallel to $[x, y]$. As a consequence of (40), $|f'(x)| \leq K$ in $V$ on almost all straight lines $L$ parallel to $[x, y]$ a.e. with respect to the linear measure on $L$. Hence we can find a sequence of line segments $[x_n, y_n] \subset V$ with the following properties: $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$; $f$ is absolutely continuous on $[x_n, y_n]$; $|f'(x)| \leq K$ a.e. on $[x_n, y_n]$ with respect to the linear measure. Then

\[
|f(x_n) - f(y_n)| \leq \int_{[x_n, y_n]} |df(z)| = \int_{[x_n, y_n]} |f'(z)dz|
\]

\[
\leq \int_{[x_n, y_n]} |f'(z)||dz| \leq K \int_{[x_n, y_n]} |dz| = K|x_n - y_n|,
\]

i.e. $|f(x_n) - f(y_n)| \leq K|x_n - y_n|$ ($n = 1, 2, \cdots$), from which (41) follows by the continuity of $f$. By the symmetry of the situations for $f$ and $f^{-1}$, we deduce the same inequality for $f^{-1}$ so that

\[
\frac{1}{K}|x - y| \leq |f(x) - f(y)| \leq K|x - y|
\]

for every $x$ and $y$ in $V$ with $[x, y] \subset V$ and $[f(x), f(y)] \subset V'$. Thus we can show the validity of (3) with respect to $\delta_{ij}$-geodesic distances $\rho$ on $V$ and $\rho'$ on $V'$ so that $f : V \rightarrow V'$ is a $K$-qi.

Combining Lemmas 37 and 39, we obtain the following result, which will be used in the final part of the proof of the main theorem 4.

42. THEOREM. Suppose that $1 \leq p < d$, $0 < K < \infty$, and $0 < \delta \leq \infty$ are arbitrarily given. Then $f, f^{-1} \in Q_p(K, \delta)$ implies that $f$ is a $K_1$-qi of $V$ onto $V'$, where $K_1 = K_1(K)$ is given by (38) so that it is independent of $\delta$.

43. Proof of the main theorem. In this section we assume that the exponent $p$ is fixed in $(1, d)$ and we choose two Riemannian manifolds $D$ and $D'$ of the same dimension $d \geq 2$ which are orientable and countable and any component of $D$ and $D'$ is not compact. The proof of the main theorem 4 consists of two parts.

First part: Assume that there exists an almost quasiisometric mapping $f$ of $D$ onto $D'$, i.e. $f$ is a homeomorphism of $D$ onto $D'$ and there exists a compact subset $E \subset D$ such that $f = f|D \setminus E$ is a $K$-quasiisometric mapping of $D \setminus E$ onto $D' \setminus E'$, where $E' = f(E)$ is a compact subset of $D'$ and $K$ a constant in $[1, \infty)$. We are to show that $f$ can be extended to a homeomorphism $f^*$ of the Royden compactification $D_p^*$ of $D$ onto that $(D')_p^*$ of $D'$. Choose an arbitrary point $\xi$ in the Royden $p$-boundary $\Gamma_p(D) = D_p^* \setminus D$. Since $D$ is dense in $D_p^*$, the point $\xi$ is an accumulation point of $D$. 
We first show that the net \((f(x_{\lambda}))\) in \(D'\) converges to a point \(\xi' \in \Gamma_{p}(D')\) for any net \((x_{\lambda})\) in \(D\) convergent to \(\xi\). Clearly \((f(x_{\lambda}))\) does not accumulate at any point in \(D'\) along with \((x_{\lambda})\) so that the cluster points of \((f(x_{\lambda}))\) are contained in \(\Gamma_{p}(D')\). Contrariwise we assume the existence of two subnets \((x_{\lambda'}^\prime)\) and \((x_{\lambda''}^\prime)\) of \((x_{\lambda})\) such that \((f(x_{\lambda'}))\) and \((f(x_{\lambda''}))\) are convergent to \(\eta'\) and \(\eta''\) in \(\Gamma_{p}(D')\), respectively, with \(\eta' \neq \eta''\). Since \(M_{p}(D')\) is dense in \(C((D')_{p}^{*})\) and forms a lattice, we can find a function \(u \in M_{p}(D')\) such that \(u \equiv 0\) in a neighborhood \(G'\) of \(E'\), \(u(\eta') = 0\), and \(u(\eta'') = 1\). Viewing \(u \in M_{p}(D' \setminus E')\), we see by Proposition 29 that \(v := u \circ f \in M_{p}(D \setminus E)\). Since \(u \equiv 0\) on the neighborhood \(G = f^{-1}(G')\) of \(E = f^{-1}(E')\), we can conclude that \(v \in M_{p}(D)\). From \(v(x_{\lambda'}) = u(f(x_{\lambda'}))\) and \(v(x_{\lambda''}) = u(f(x_{\lambda''}))\) it follows that \(v(\xi') = u(\eta') = 0\) and \(v(\xi'') = u(\eta'') = 1\), which is a contradiction.

We next show that the nets \((f(x_{\lambda}))\) and \((f(y_{\lambda}))\) in \(D'\) converge to a point in \(\Gamma_{p}(D')\) for any two nets \((x_{\lambda})\) and \((y_{\lambda})\) convergent to \(\xi \in \Gamma_{p}(D)\). In fact, let \((x_{\lambda})\) be a net convergent to \(\xi\) such that \((x_{\lambda})\) contains \((x_{\lambda'})\) and \((y_{\lambda''})\) as its subnets. Then we see that \(\lim f(x_{\lambda'}) = \lim f(x_{\lambda}) = \lim f(x_{\lambda})\). Hence we have shown that \(f^*(\xi) := \lim_{\lambda \rightarrow \eta} f(x_{\lambda}) \in \Gamma_{p}(D')\) for any \(\xi \in \Gamma_{p}(D)\). On setting \(f^* = f\) on \(D\), we see that \(f^*\) is a continuous mapping of \(D_{p}^{*}\) onto \((D')_{p}^{*}\). The uniqueness of \(f^*\) on \(D_{p}^{*}\) is a consequence of the denseness of \(D\) in \(D_{p}^{*}\). Similarly we can show that \(f^{-1}\) can also be uniquely extended to a continuous mapping \((f^{-1})^*\) of \((D')_{p}^{*}\) onto \(D_{p}^{*}\). Since \((f^{-1})^* \circ f^*\) and \(f^* \circ (f^{-1})^*\) are identities on \(D_{p}^{*}\) and \((D')_{p}^{*}\), respectively, as the unique extensions of \(id. : D \rightarrow D\) and \(id. : D' \rightarrow D'\), respectively, we see that \(f^*\) is a homeomorphism of \(D_{p}^{*}\) onto \((D')_{p}^{*}\) which is the unique extension of \(f : D \rightarrow D'\).

\(\Box\)

Second part : Suppose the existence of a homeomorphism \(f^*\) of \(D_{p}^{*}\) onto \((D')_{p}^{*}\). We are to show that \(f := f^*|D\) is an almost quasiisometric mapping of \(D\) onto \(D'\), which is the essential part of this note.

Choose an arbitrary point \(x \in D\). Since \(x \in G_{\delta}\), \(f^*(x) \in (D')_{p}^{*}\) is also \(G_{\delta}\) so that \(f^*(x) \in D'\) by Corollary 18. Thus we have shown that \(f^*(D) \subset D'\). Similarly we can conclude that \((f^*)^{-1}(D') \subset D\). These show that \(f^*(D) = D'\) and therefore \(f := f^*|D\) is a homeomorphism of \(D\) onto \(D'\). We are to show that \(f\) is an almost quasiisometric mapping of \(D\) onto \(D'\).

We fix a family \(V = \{V\}\) of open sets \(V\) in \(D\) with the following properties: \(V\) is contained in a 2-domain \(U_{V}\) in \(D\) and \(V' := f(V)\) is contained in the 2-domain \(U_{V'} = f(U_{V})\) in \(D'\); \(\cup_{V \in V'} = D\). This is possible since the family of 2-domains forms a base of open sets on any Riemannian manifold and \(f : D \rightarrow D'\) is a homeomorphism. We set \(V' := \{V' : V' = f(V) (V \in V)\}\), which enjoys the same properties as \(V\) does. We also fix an exhaustion \((\Omega_{n})_{n \geq 1}\) of \(D\), i.e. \(\Omega_{n}\) is a relatively compact open subset of \(D (n = 1, 2, \cdots)\), \(\Omega_{n} \subset \Omega_{n+1} (n = 1, 2, \cdots)\), and \(\cup_{n \geq 1} \Omega_{n} = D\). Then \((\Omega_{n}')_{n \geq 1}\) with \(\Omega_{n}' := f(\Omega_{n}) (n = 1, 2, \cdots)\) also forms an exhaustion of \(D'\). We set \(D_{n} := D \setminus \overline{\Omega_{n}}\) and \(D_{n}' := D' \setminus \overline{\Omega_{n}'}\) \((n = 1, 2, \cdots)\). Then \((D_{n})_{n \geq 1} ((D_{n}')_{n \geq 1}, \text{resp.})\) is a decreasing sequence of open sets \(D_{n}\).
if of of for fact of contradicting proof Assertion the tensor we for not that every case metric every those we Then see that this assertion maintain a a where where inclusion every an resp.) for Take an arbitrary n \{1, 2, \cdots\}. If f \in Q_p(2^{n+p-1}, 2^{-m}; V \cap D_m, V' \cap D_n') (f^{-1} \in Q_p(2^{m+p-1}, 2^{-m}; V' \cap D_n', V \cap D_m), resp.) for every V \in \mathcal{V} with V \cap D_m \neq \emptyset (so that V' \cap D_n' \neq \emptyset), where V' = f(V) and V' \cap D_n' = f(V) \cap f(D_n) = f(V \cap D_n), then we write
\[ f \in (n) \quad (f^{-1} \in (n), \text{ resp.}). \]
Hence, for example, f \notin (n) means that there exists a V \in \mathcal{V} with V \cap D_n \neq \emptyset such that f \notin Q_p(2^{n+p-1}, 2^{-n}; V \cap D_n, V' \cap D_n'). We maintain

44. Assertion. If f \in (n) (f^{-1} \in (n), \text{ resp.}) for some n, then f \in (m) (f^{-1} \in (m), \text{ resp.}) for every m \geq n.

In fact, f \in (n) assures that f \in Q_p(2^{n+p-1}, 2^{-n}; V \cap D_n, V' \cap D_n') for every V \in \mathcal{V} with V \cap D_n \neq \emptyset. Choose any V \in \mathcal{V} with V \cap D_m \neq \emptyset. Since D_m \subset D_n, V \cap D_n \neq \emptyset along with V \cap D_m \neq \emptyset and therefore f \in Q_p(2^{m+p-1}, 2^{-m}; V \cap D_m, V' \cap D_n'). In view of the fact that 2^{m+p-1} \leq 2^{m+p-1} and 2^{-n} \geq 2^{-m}, we have the inclusion relation Q_p(2^{m+p-1}, 2^{-m}; V \cap D_m, V' \cap D_n') \supset Q_p(2^{n+p-1}, 2^{-n}; V \cap D_n, V' \cap D_n') so that f \in Q_p(2^{m+p-1}, 2^{-m}; V \cap D_m, V' \cap D_n'), i.e. f \in (m), which completes the proof of Assertion 44. Next we assert

45. Assertion. If f \in (n) and f^{-1} \in (n) for some n, then f = f|D_n is a qi of D_n onto D_n'.

Indeed, by Theorem 42, we see that f : (V \cap D_n, \delta_{ij}) \longrightarrow (V' \cap D_n', \delta_{ij}) is a K_1-qi with K_1 = K_1(2^{n+p-1}) (cf. (38) in Lemma 37). Clearly id. : (V \cap D_n, g_{ij}) \longrightarrow (V \cap D_n, \delta_{ij}) and id. : (V' \cap D_n', \delta_{ij}) \longrightarrow (V' \cap D_n', g_{ij}') are \sqrt{2}-qi, where (g_{ij}') is the metric tensor on D'. Therefore, as the suitable composition of these maps above, we see that f : (V \cap D_n, g_{ij}) \longrightarrow (V' \cap D_n', g_{ij}') is a 2K_1-qi. Since this is true for every V \in \mathcal{V} with V \cap D_n \neq \emptyset and \bigcup_{V \in \mathcal{V}} V = D \supset D_n, we can conclude that f : D_n \longrightarrow D_n' is a 2K_1-qi. The proof of Assertion 45 is thus complete.

To complete the proof of this second part it is sufficient to show that f : D_n \longrightarrow D_n' is a qi for some n. We prove it by contradiction. Contrariwise suppose that f : D_n \longrightarrow D_n' is not qi for every n = 1, 2, \cdots. Then we maintain that either f \notin (n) for every n or f^{-1} \notin (n) for every n. In fact, if f \notin (n) for every n, then we are done. Otherwise, there is a k with f \in (k). Then by Assertion 44 we have f \in (n) for every n \geq k. In this case we must have f^{-1} \notin (n) for every n and the assertion is assured. To see this assume that f^{-1} \in (l) for some l. Then f^{-1} \in (n) for every n \geq l again by Assertion 44. Then f \in (k \cup l) and f^{-1} \in (k \cup l). By Assertion 45 we see that f is a qi of D_{k\cup l} onto D_{k\cup l}, contradicting our assumption. On interchanging the roles of f and f^{-1} (and thus those of D and D') if
necessary, we can assume that

\[ f \not\in (n) \quad (n = 1, 2, \cdots), \]

from which we will derive a contradiction.

The fact that \( f \not\in (1) \) implies the existence of a 2-domain \( V \in \mathcal{V}_{D_1} \) such that \( f \not\in Q_p(2^{1+p-1}, 2^{-1}; V, f(V)) \). We can then find a spherical ring \( S_1 \subset V(\subset D_1) \) such that

\[ \text{cap}_p S_1 < 2^{-1}, \quad \text{cap}_p, f(S_1) > 2^{1+p-1}\text{cap}_p S_1. \]

Here \( \text{cap}_p S_1 \) means \( \text{cap}_p(S_1, \delta_{ij}) \). We set \( n_1 := 1 \). Let \( n_2 \) be the least integer such that \( n_2 \geq n_1 + 1 \) (and hence \( D_{n_1+1} \supset D_{n_2} \)) and \( \overline{D_{n_2}} \cap \overline{S_{n_1}} = \emptyset \). Since \( f \not\in (n_2) \), there exists a \( V \in \mathcal{V}_{D_{n_2}} \) with \( f \not\in Q_p(2^{n_2+p-1}, 2^{-n_2}; V, f(V)) \). Hence we can find a spherical ring \( S_{n_2} \subset V(\subset D_{n_2}) \) such that

\[ \text{cap}_p S_{n_2} < 2^{-n_2}, \quad \text{cap}_p, f(S_{n_2}) > 2^{n_2+p-1}\text{cap}_p S_{n_2}, \]

where \( \text{cap}_p S_{n_2} \) means \( \text{cap}_p(S_{n_2}, \delta_{ij}) \). Repeating this process we can construct a sequence \((S_{n_k})_{k \geq 1}\) of spherical rings \( S_{n_k} \) with the following properties: \( n_k + 1 \leq n_{k+1} \); \( S_{n_k} \subset D_{n_k} \); \( \overline{S_{n_k}} \cap \overline{S_{n_l}} = \emptyset (k \neq l); \)

\[ \text{cap}_p S_{n_k} < 2^{-n_k}, \quad \text{cap}_p, f(S_{n_k}) > 2^{n_k+p-1}\text{cap}_p S_{n_k} \quad (k = 1, 2, \cdots). \]

Fix a \( k \) and set \( T = S_{n_k} \). Since it is a spherical ring in a 2-domain \((U_{V_{n_k}}, x)\) and contained in \( V_{n_k} \), \( T \) has a representation \( T = \{x : a < |x - P| < b\} \), where \( P \in V_{n_k} \) and \( 0 < a < b < \infty \). Let \( l = [(2^{-n_k}/\text{cap}_p T)^{1/(p-1)}] > 0 \) where \([ \cdot ]\) is the Gaussian symbol, which means that

\[ (47) \quad p^{-1} \leq \frac{2^{-n_k}}{\text{cap}_p T} < (l + 1)^{p-1} \leq 2^{p-1} p^{-1}. \]

Using the notation \( q = (p - d)/(p - 1) \) (cf. (13)) we set

\[ t_j := \left( \frac{(l-j)aq + jbj}{l} \right)^{\frac{1}{q}} \quad (j = 0, 1, \cdots, l). \]

We divide the ring \( T \) into \( l \) small spherical rings \( T_j \) given by

\[ T_j := \{x : t_{j-1} < |x - P| < t_j\} \quad (j = 0, 1, \cdots, l). \]

By (13) we have \( \text{cap}_p T = \text{cap}_p (T, \delta_{ij}) = \omega_d((b^q - a^q)/q)^{1-p} \). Similarly

\[ \text{cap}_p T_j = \omega_d \left( \frac{t_j^q - t_{j-1}^q}{q} \right)^{1-p}. \]
\[
\omega_d \left( \frac{(l-j)a^q + j b^q}{l} - \frac{(l-j+1)a + q(j-1)b^q}{ql} \right)^{1-p} = \omega_d \left( \frac{b^q - a^q}{q} \right) l^{p-1} = l^{p-1} \text{cap}_p T,
\]
i.e. we have shown that \(	ext{cap}_p T_j = l^{p-1} \text{cap}_p T\). Therefore we have the following identity for the subdivision \(\{T_j\}_{1 \leq j \leq l}\) of \(T\):

\[
(48) \quad \sum_{j=1}^{l} \left( \text{cap}_p T_j \right)^{1 \over 1-p} = \left( \text{cap}_p T \right)^{1 \over 1-p}.
\]

Concerning the induced subdivision \(\{f(T_j)\}\) of \(f(T)\), the general inequality (10) implies the inequality

\[
(49) \quad \sum_{j=1}^{l} \left( \text{cap}_p f(T_j) \right)^{1 \over 1-p} \leq \left( \text{cap}_p f(T) \right)^{1 \over 1-p}.
\]

Now suppose that \(\text{cap}_p f(T_j) \leq 2^{n_k+p-1} \text{cap}_p T_j\) for every \(1 \leq j \leq l\). Then \(\left( \text{cap}_p f(T_j) \right)^{1/(1-p)} \geq 2^{(n_k+p-1)/(1-p)} \left( \text{cap}_p T_j \right)^{1/(1-p)}\) for every \(1 \leq j \leq l\). By using (49) and (48) we deduce

\[
\left( \text{cap}_p f(T) \right)^{1 \over 1-p} \geq \sum_{j=1}^{l} \left( \text{cap}_p f(T_j) \right)^{1 \over 1-p} \geq 2^{n_k+p-1} \sum_{j=1}^{l} \left( \text{cap}_p T_j \right)^{1 \over 1-p} = 2^{n_k+p-1} \left( \text{cap}_p T \right)^{1 \over 1-p},
\]
which means that \(\text{cap}_p f(T) \leq 2^{n_k+p-1} \text{cap}_p T\). This contradicts (46) since \(T = S_{n_k}\). Therefore there must exist a number \(j_0 \in \{1, \ldots, l\}\) such that

\[
(50) \quad \text{cap}_p f(T_{j_0}) > 2^{n_k+p-1} \text{cap}_p T_{j_0}.
\]

We now set \(R_k := T_{j_0}\). By (47) we have \(l^{p-1} \text{cap}_p T \leq 2^{-n_k} \leq 2^{p-1} l^{p-1} \text{cap}_p T\). Since \(l^{p-1} \text{cap}_p T = \text{cap}_p T_{j_0} = \text{cap}_p R_k\), we see that

\[
\text{cap}_p R_k \leq 2^{-n_k} \leq 2^{p-1} \text{cap}_p R_k.
\]
This is equivalent to \(\text{cap}_p R_k < 2^{-n_k} < 2^{-k}\) (since \(n_k > k\)) and \(\text{cap}_p R_k \geq 2^{-n_k-p+1}\). The latter inequality with (50) implies that \(\text{cap}_p f(R_k) > 2^{n_k+p-1} \text{cap}_p R_k \geq 2^{n_k+p-1} \geq 1\). By (46), \(\text{cap}_p (R_k, g_{ij}) < 2^{(d+p)/2} \cdot 2^{-k}\) and \(\text{cap}_p (f(R_k), g_{ij}) > 2^{(d+p)/2}\).

We have thus constructed an admissible sequence \((R_k)_{k \geq 1}\) of rings \(R_k\) in \(D\) in the sense of §8 (cf. Lemma 15) such that \(\text{cap}_p R_k = \text{cap}_p (R_k, g_{ij})\) and \(\text{cap}_p f(R_k) = \text{cap}_p (f(R_k), g_{ij})\) satisfy

\[
(51) \quad \text{cap}_p R_k < 2^{(d+p)/2} \cdot 2^{-k} \quad \text{and} \quad \text{cap}_p f(R_k) > 2^{(d+p)/2}
\]
for every $k = 1, 2, \cdots$. Let $C_{k1}$ be the inner part of $R_k^c = D \setminus R_k$ and we set

$$X := \bigcup_{k=1}^{\infty} C_{k1} \quad \text{and} \quad Y := \bigcap_{k=1}^{\infty} (D \setminus (R_k \cup C_{k1}))$$

as in §8 (cf. Lemma 15). The first inequality in (51) implies that

$$\sum_{k=1}^{\infty} \text{cap}_p R_k < \sum_{k=1}^{\infty} 2^{\frac{d+p}{2}} \cdot 2^{-k} = 2^{\frac{d+p}{2}} < \infty$$

and therefore Lemma 15 assures that

$$(\text{cl}(X;D^*_p)) \cap (\text{cl}(Y;D^*_p)) = \emptyset.$$ 

Due to the fact that $f^*$ is a homeomorphism of $D^*_p$ onto $(D')^*_p$, we see that

$$f^*(\text{cl}(X;D^*_p)) \cap f^*(\text{cl}(Y;D^*_p)) = f^*(\emptyset) = \emptyset.$$ 

Since again $(f(R_k))_{k \geq 1}$ is an admissible sequence of rings $f(R_k)$ on $D'$, the above relation must imply by Lemma 15 that $\sum_{k=1}^{\infty} \text{cap}_p f(R_k) < \infty$. However the second inequality in (51) implies that

$$\sum_{k=1}^{\infty} \text{cap}_p f(R_k) \geq \sum_{k=1}^{\infty} 2^{\frac{d+p}{2}} = \infty,$$

which is a contradiction. This comes from the erroneous assumption that $f : D_n \rightarrow D'_n$ is not a qi for every $n = 1, 2, \cdots$, and thus we have established the existence of an $n$ such that $f = f|D_n$ is a qi of $D_n$ onto $D'_n$. The second part of the proof for the main theorem 4 is herewith complete. \hfill \square

References


