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On some conditions of starlikeness

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Abstract

Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be analytic in the unit disk U. Yuan Chun Fang proved that

$$|\frac{f''(z)}{f'(z)}| < m \Longrightarrow Re\frac{zf'(z)}{f(z)} > 0 \quad (z \in U),$$

where $m(= 2.83\cdots)$ is the best possible. In this paper, we generalize this theorem.

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2}\cdots\cdots \quad (p \in N), \tag{1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$, and M_p denote the class of functions

$$f(z) = z^{-p} + a_{-p+1}z^{-p+1} + a_{-p+2}z^{-p+2}\cdots\cdots \quad (p \in N), \tag{2}$$

which are analytic in punctured disk $U - \{0\}$. Then we denote two classes of starlike functions as follows:

$$A_p^* = \{f \in A_p : \Re\frac{zf'(z)}{f(z)} > 0,\ z \in U\}, \tag{3}$$

$$M_p^* = \{f \in M_p : \Re\frac{zf'(z)}{f(z)} < 0,\ z \in U\}. \tag{4}$$

For the class A_1, Singh and Singh [4] showed the following theorem.

Theorem A *Let $f(z) \in A_1$, then*

(5) $$\left|\frac{f''(z)}{f'(z)}\right| < \frac{3}{2} \quad (z \in U) \Longrightarrow f(z) \in A_1^*.$$

He used Jack's Lemma. Mocanu [3,p.338] showed

Theorem B *Let $g(z) = \frac{e^{\lambda z}-1}{\lambda}$.*

(6) $$g(z) \in A_1^* \Longleftrightarrow |\lambda| \leq m = 2.8329\cdots.$$

Where m is the least positive solutin of the following equation

(7) $$\cos\sqrt{x^2-1} + \sqrt{x^2-1}\sin\sqrt{x^2-1} - \frac{1}{e} = 0.$$

Miller and Mocanu [2] proved, using their theory of first order differential subordination, that

Theorem C *Let $f(z) \in A_1$.Then*

(8) $$\left|\frac{f''(z)}{f'(z)}\right| < 2 \Longrightarrow f(z) \in A_1^*.$$

In the same article [2], they posed the interesting question of finding the maximum value of k for which

(9) $$\left|\frac{f''(z)}{f'(z)}\right| < k \Longrightarrow f(z) \in A_1^*.$$

From the above two theorems, $2 \leq k \leq m$. Some Mathematicians improved the lower bound of k. And recently, Yuan Chun Fang [4] showed

Theorem D *Let $f(z) \in A_1$. Then*

(10) $$\left|\frac{f''(z)}{f'(z)}\right| < m \quad (z \in U) \Longrightarrow f(z) \in A_1^*.$$

The result is sharp, with the extremal function

(11) $$G(z) = \frac{e^{mz}-1}{m}.$$

The purpose of this paper is to obtain simillar theorems for A_p and M_p.

Theorem 1 *If $f(z) \in A_p$ satisfies*

$$|\frac{f''(z)}{f'(z)} - \frac{p-1}{p}\frac{f'(z)}{f(z)}| \leq m \quad (z \in U), \tag{12}$$

then f belongs to A_p^*. The result is sharp, with the extremal function

$$G_1(z) = (\frac{e^{mz}-1}{m})^p. \tag{13}$$

Theorem 2 *If $f(z) \in M_p$ satisfies*

$$|\frac{f''(z)}{f'(z)} - \frac{p+1}{p}\frac{f'(z)}{f(z)}| \leq m \quad (z \in U), \tag{14}$$

then f belongs to M_p^*. The result is sharp, with the extremal function

$$G_2(z) = (\frac{m}{e^{mz}-1})^p. \tag{15}$$

2. Proof of Theorem 1

We use the following lemma due to Miller and Mocanu [2, Theorem 2].

Lemma *let h be convex in U and θ and ϕ be analytic in a domain D. Let p be analytic in U, with $p(0) = h(0) = \theta(p(0))$ and $p(U) \subset D$. If the differential equation*

$$\theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{16}$$

has a univalent solution in U that satisfies $q(0) = h(0)$ and

$$\theta(q(z)) \prec h(z), \tag{17}$$

then the relation

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec h(z) \tag{18}$$

implies $p(z) \prec q(z)$. The function q is the best dominant of (18).

Suppose $f(z) \in A_p^*$ satisfies (12), then we have

$$\frac{zf''(z)}{f'(z)} - \frac{p-1}{p}\frac{zf'(z)}{f(z)} \prec mz. \tag{19}$$

Let put

$$p(z)=\frac{1}{p}\frac{zf'(z)}{f(z)},\qquad q(z)=\frac{1}{p}\frac{zG_1'(z)}{G_1(z)}$$

$$h(z)=1+mz,\qquad \theta(z)=z,\quad and\quad \phi(z)=\frac{1}{z}.$$

Then we have

$$q(z)=\frac{zG'(z)}{G(z)}=m\frac{ze^{mz}}{e^{mz}-1}. \tag{20}$$

$$q(z)+\frac{zq'(z)}{q(z)}=1+mz, \tag{21}$$

and

$$p(z)+\frac{zp'(z)}{p(z)}=1+\frac{zf''(z)}{f'(z)}-\frac{p-1}{p}\frac{zf'(z)}{f(z)}. \tag{22}$$

From (19) and (22) we obtain (18), and from (21) we obtain (16). It yields $p(z)\prec q(z)$, and so $\frac{zf'(z)}{f(z)}\prec\frac{zG'(z)}{G(z)}$. Therefore, from Theorem B we obtain that

$$\frac{zf'(z)}{f(z)}>0\quad (z\in U).$$

Concernig the exremal function, we have

$$\frac{zG_1''(z)}{G_1'(z)}-\frac{p-1}{p}\frac{G_1f'(z)}{G_1(z)}=m.$$

This implies that $G_1(z)$ is exremal.

Proof of Theorem 2 is similar, so we omit.

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