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Chaotic Translation Semigroups of Linear Operators

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1 Introduction

Hypercyclic or chaotic operators are consistent with topologically transitive or chaotic, respectively in topological spaces defined by Devaney [6]. The property of hypercyclic and chaotic operators has been studied by some people [1, 2, 7, 8, 10, 11]. C. Read has developed the theory of hypercyclic and chaotic bounded linear operators in connection with the invariant subspace problem of Hilbert spaces [10]. W. Desch, W. Schappacher and G. F. Webb gave a necessary and sufficient condition for a semigroup to be hypercyclic in a separable Banach space [2]. The theory of hypercyclic semigroups is applied to the scattering theory for a linear transport equation [3]. Concerning to the chaotic semigroup, a sufficient condition for a semigroup to be chaotic is given in a separable Banach space [2]. In [9], chaotic semigroups are associated with the idea of exactness and applied to partial differential equations.

In this paper, we give necessary and sufficient conditions for the translation semigroup to be chaotic in weighted function spaces L^p_ρ and $C_{0,\rho}$. We also investigate properties of orbits of the translation semigroup when the set of periodic points is dense in weighted function spaces, and give an example which shows that some solutions of partial differential equations become chaotic semigroups. We shall define hypercyclic and chaotic semigroups.

Definition 1. Let X be a Banach space and $\{T(t)\}$ be a strongly continuous semigroup in X. The semigroup $\{T(t)\}$ is called *hypercyclic* if there exists $x \in X$ such that the set $\{T(t)x \mid t > 0\}$ is dense in X. The semigroup $\{T(t)\}$ is called *chaotic* if $\{T(t)\}$ is hypercyclic and the set of periodic points $X_p = \{x \in X \mid \exists t > 0 \ s.t. \ T(t)x = x\}$ is dense in X.

We define an admissible weight function in order to construct weighted function spaces. Let I be $(-\infty, \infty)$ or $I = [0, \infty)$. By an admissible weight function in I we mean a measurable function $\rho : I \to \mathbb{R}$ satisfying the following conditions:

(i) $\rho(\tau) > 0 \quad for \ all \ \tau \in I;$

(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq Me^{\omega t}\rho(t + \tau)$ holds for all $\tau \in I$ and all $t > 0$.

With an admissible weight function, we construct the following function spaces.

Definition 2.

$$L^p_\rho(I, \mathbb{C}) = \left\{ u : I \to \mathbb{C} \mid u \ measurable, \int_I |u(\tau)|^p \rho(\tau) \ d\tau < \infty \right\}$$

$$\text{with } \|u\| = \left(\int_I |u(\tau)|^p \rho(\tau) \ d\tau \right)^{\frac{1}{p}},$$

$$C_{0,\rho}(I, \mathbb{C}) = \left\{ u : I \to \mathbb{C} \mid u \ continuous, \lim_{\tau \to \pm\infty} \rho(\tau) u(\tau) = 0 \right\}$$

$$\text{with } \|u\| = \sup_{\tau \in I} |u(\tau)| \rho(\tau).$$

Let X be $C_{0,\rho}(I)$ or $L^p_\rho(I)$ and $X_{0,0}$ be the set of all functions with compact support in X. Then $X_{0,0}$ is dense in X. We consider the translation semigroup $\{T(t)\}_{t \geq 0}$ in X as follows:

$$[T(t)u](\tau) = u(\tau + t) \quad \text{for } u \in X.$$

A condition for this translation semigroup to be hypercyclic is given in [2].

Theorem A [2]. *Let X be $L^p_\rho(I)$ or $C_{0,\rho}(I)$ with an admissible weight function ρ. Then the following* (1) *and* (2) *are equivalent:*

(1) *The translation semigroup $\{T(t)\}$ in X is hypercyclic;*

(2) (i) *If $I = [0, \infty)$, then $\liminf_{t\to\infty} \rho(t) = 0$ holds.*

(ii) *If $I = (-\infty, \infty)$, then for each $\theta \in \mathbb{R}$ there exists a sequence $\{t_j\}_{j=1}^\infty$ of positive real numbers such that*

$$\lim_{j\to\infty} \rho(t_j + \theta) = \lim_{j\to\infty} \rho(-t_j + \theta) = 0.$$

2 Chaotic translation semigroups

We shall give necessary and sufficient conditions that translation semigroups are chaotic in weighted function spaces and also explain an result about the property of the orbit of translation semigroups when the set of periodic points is dense in weighted function spaces. Though the conditions for the translation semigroup to be hypercyclic depend on whether $I = (-\infty, \infty)$ or $I = [0, \infty)$, the condition to be chaotic depends on whether the space is L^p_ρ or $C_{0,\rho}$. For the case of L^p_ρ, we have the following theorem.

Theorem 2.1. *Let $I = (-\infty, \infty)$ (resp. $I = [0, \infty)$) and let X be $L^p_\rho(I)$. Then the translation semigroup $\{T(t)\}$ is chaotic if and only if for all $\epsilon > 0$ and for all $l > 0$, there exist $P > 0$ such that*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \rho(l + nP) < \epsilon. \ (resp. \ \sum_{n=1}^{\infty} \rho(l + nP) < \epsilon).$$

Proof. The condition is necessary. Suppose $\{T(t)\}$ is chaotic and ρ satisfies that $\rho(\tau) \le Me^{\omega t}\rho(\tau + t)$ for $\tau \in I$ and $t > 0$. Take $\epsilon > 0$, $l > 0$ and $z \in X$ such that $\|z\| = 1$ and $\mathrm{supp}(z) \subset [l, l + \theta]$ for some $\theta > 0$.

Take positive ϵ' satisfying the following condition:

$$(\epsilon')^p < \min\left\{\frac{e^{-2\omega\theta}\epsilon}{2^p M^2 \rho(l+\theta)}, \frac{1}{2^p}\right\}.$$

Since X_p is dense in X, there exists $v \in X_p$ such that

$$\|z - v\| < \epsilon'.$$

For $v \in X_p$, there exists $P > 0$ such that

$$v = T(nP)v \quad \text{for all } n \in \mathbb{N}.$$

So we obtain $\|z - T(nP)v\| < \epsilon'$. By replacing P with mP for enoughly large $m \in \mathbb{N}$, we can choose P such as $P > \theta$. The following assertion holds for each positive integer n. Let w_n be the restriction of v to the interval $[l + nP, l + nP + \theta]$. Then

$$\text{supp}(T(nP)w_n) \subset [l, l+\theta]$$

and

$$\|z - T(nP)w_n\| < \epsilon'$$

hold by relations $\text{supp}(z) \subset [l, l+\theta]$ and $\|z - T(nP)v\| < \epsilon'$. So

$$\|z\| - \|T(nP)w_n\| < \epsilon'.$$

$$\begin{aligned}\|T(nP)w_n\| &> 1 - \epsilon' \ (\|z\| = 1)\\ &> 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

Next we calculate $\|T(nP)w_n\|$. From the necessary and sufficient condition for hypercyclicity of $\{T(t)\}$ in Theorem A and the property of an admissible weight function, ω must be positive.

So we obtain the following inequality:

$$\begin{aligned}
\|T(nP)w_n\|^p &= \int_0^\infty \rho(\tau)\cdot|T(nP)w_n(\tau)|^p\,d\tau\\
&= \int_l^{l+\theta} \rho(\tau)\cdot|w_n(\tau+nP)|^p\,d\tau\\
&= \int_{l+nP}^{l+nP+\theta} \rho(\tau-nP)\cdot|w_n(\tau)|^p\,d\tau\\
&\le \int_{l+nP}^{l+nP+\theta} Me^{\omega\theta}\cdot\rho(l+\theta)\cdot|w_n(\tau)|^p\,d\tau\\
&= M\cdot e^{\omega\theta}\cdot\rho(l+\theta)\int_{l+nP}^{l+nP+\theta}|w_n(\tau)|^p\,d\tau.
\end{aligned}$$

Therefore

$$\int_{l+nP}^{l+nP+\theta}|v(\tau)|^p\,d\tau = \int_{l+nP}^{l+nP+\theta}|w_n(\tau)|^p d\tau \ge \frac{e^{-\omega\theta}}{2^pM\rho(l+\theta)}.$$

If $n\in\mathbb{Z}^-$, then

$$\int_{l+nP}^{l+nP+\theta}|v(\tau)|^p\,d\tau = \int_l^{l+\theta}|v(\tau)|^p\,d\tau \ge \frac{e^{-\omega\theta}}{2^pM\rho(l+\theta)}$$

from $T(nP)v=v$. Since $\operatorname{supp}(z)\subset[l,l+\theta]$,

$$\begin{aligned}
(\epsilon')^p &> \|z-v\|^p\\
&\ge \sum_{n\in\mathbb{Z}\setminus\{0\}}\int_{l+nP}^{l+nP+\theta}\rho(\tau)|v(\tau)|^p\,d\tau\\
&\ge \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{M}e^{\omega\theta}\rho(l+nP)\int_{l+nP}^{l+nP+\theta}|v(\tau)|^p\,d\tau\\
&\ge \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{e^{-2\omega\theta}\cdot\rho(l+nP)}{2^pM^2\rho(l+\theta)}.
\end{aligned}$$

So we obtain

$$\sum_{n\in\mathbb{Z}\setminus\{0\}}\rho(l+nP) < (\epsilon')^p\cdot e^{2\omega\theta}\cdot 2^{p-1}M^2\rho(l+\theta) < \epsilon.$$

The condition is sufficient. It is clear that $\{T(t)\}$ is hypercyclic by Theorem A. So we only have to show that the set of periodic points X_p is dense in X.

Since the set $X_{0,0}$ of all the functions with compact support is dense in X, we shall show that X_p is dense in $X_{0,0}$. Take $\epsilon > 0$ and $z \in X_{0,0}$. Then there exists $l > 0$ such that

$$\mathrm{supp}(z) \subset [-l, l].$$

By the condition and the property of an admissible weight function, ω must be positive.

So we obtain that for all $\sigma \in I$

$$\frac{1}{M} e^{-2\omega l} \rho(\sigma) \leq \rho(\tau) \leq M e^{2\omega l} \rho(\sigma + 2l) \quad for\ \tau \in [\sigma, \sigma + 2l].$$

Take ϵ' such as

$$0 < \epsilon' < \frac{e^{-4\omega l} \rho(-l)}{M^2 \|z\|} \epsilon.$$

From the assumption, there exists $P > 0$ such that

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \rho(l + nP) < \epsilon'.$$

By replacing P with mP for enoughly large $m \in \mathbb{N}$, we can obtain $P > 2l$.

We shall construct v_p in the following way:

$$v_p = \sum_{n \in \mathbb{Z}} z(\tau - nP).$$

Then clearly $T(P) v_p = v_p$, so $v_p \in X_p$.

We calculate $\|z - v_p\|$.

$$\begin{aligned} \|z - v_p\| &= \left\| \sum_{n \in \mathbb{Z} \setminus \{0\}} z(\tau - nP) \right\| \\ &\leq \sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{-\infty}^{\infty} \rho(\tau) \cdot |z(\tau - nP)| \, d\tau \\ &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{-l+nP}^{l+nP} \rho(\tau) \cdot |z(\tau - nP)| \, d\tau \\ &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{-l+nP}^{l+nP} \rho(\tau - nP) \cdot \frac{\rho(\tau)}{\rho(\tau - nP)} \cdot |z(\tau - nP)| \, d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{M^2 e^{4\omega l}\rho(l+nP)}{\rho(-l)}\cdot\|z\| \\
&= \frac{M^2 e^{4\omega l}\|z\|}{\rho(-l)}\left\{\sum_{n\in\mathbb{Z}\setminus\{0\}}\rho(l+nP)\right\} \\
&\leq \epsilon.
\end{aligned}
$$

(In the case of $I = [0,\infty)$, replace $-l$, $2l$, $\mathbb{Z}$ and $\mathbb{Z}\setminus\{0\}$ with 0, l, $\mathbb{Z}^+$ and $\mathbb{N}$.)

So X_p is dense in X. Therefore $\{T(t)\}$ is chaotic. □

For the case of $C_{0,\rho}$, we have the following theorem.

Theorem 2.2. *Let $I = (-\infty,\infty)$ (resp. $I = [0,\infty)$) and let X be $C_{0,\rho}(I)$. Then the following assertions are equivalent:*

(i) *the translation semigroup $\{T(t)\}$ in X is chaotic;*

(ii) *for all $\epsilon > 0$ and for all $l > 0$, there exists $P > 0$ such that*

$$\rho(l+nP) < \epsilon \quad \text{for all } n \in \mathbb{Z}\setminus\{0\} \ (\text{resp. } n \in \mathbb{N});$$

(iii) *there exists $\{l_i\}_{i=1}^{\infty} \subset \mathbb{R}^+$ whose limit is infinity, such that for all $\epsilon > 0$ and for all $i \in \mathbb{N}$, there exists $P > 0$ such that*

$$\rho(l_i+nP) < \epsilon \quad \text{for all } n \in \mathbb{Z}\setminus\{0\} (\text{resp. } n \in \mathbb{N}).$$

Proof. (i) $\Rightarrow$ (ii): Suppose $\{T(t)\}$ is chaotic and ρ satisfies that $\rho(\tau) \leq Me^{\omega t}\rho(\tau+t)$ for $\tau \in I$ and $t > 0$. Take $\epsilon > 0$, $l > 0$ and $z \in X$ with compact support such that $z(l) \neq 0$. Take ϵ' such as

$$0 < \epsilon' < \frac{\epsilon\cdot\rho(l)\cdot|z(l)|}{\rho(l)+\epsilon}.$$

Because X_p is dense in X, there exists $v \in X_p$ such that

$$\|z - v\| < \epsilon'.$$

For $v \in X_p$, there exists $P > 0$ such that

$$v = T(nP)v \qquad \text{for all } n \in \mathbb{N}.$$

Then

$$\begin{aligned} \epsilon' > \|z - v\| &= \sup_{\tau \in I} \rho(\tau)|z(\tau) - v(\tau)| \\ &> \sup_{\tau \in I} \rho(\tau)\left(|z(\tau)| - |v(\tau)|\right) \\ &> \rho(l)\left(|z(l)| - |v(l)|\right). \end{aligned}$$

So

$$|v(l)| > |z(l)| - \frac{\epsilon'}{\rho(l)}.$$

By replacing P with mP for enoughly large $m \in \mathbb{N}$, we can choose $P > 0$ such that $l \pm nP$ (resp. $l + nP$)$\notin$ supp(z) for all $n \in \mathbb{Z} \setminus \{0\}$. Then we obtain the following inequalities for each $n \in \mathbb{Z} \setminus \{0\}$(resp. $n \in \mathbb{N}$).

$$\begin{aligned} \epsilon' > \|z - v\| &= \sup_{\tau \in I} \rho(\tau)|z(\tau) - v(\tau)| \\ &\geq \rho(l + nP) \cdot |v(l + nP)| \\ &= \rho(l + nP) \cdot |v(l)| \\ &> \rho(l + nP) \cdot \left(|z(l)| - \frac{\epsilon'}{\rho(l)}\right). \end{aligned}$$

So

$$\begin{aligned} \rho(l + nP) &< \epsilon' / \left(|z(l)| - \frac{\epsilon'}{\rho(l)}\right) \\ &< \epsilon. \end{aligned}$$

Therefore for all $l > 0$ and for all $\epsilon > 0$, there exists $P > 0$ such that

$$\rho(l + nP) < \epsilon \qquad \text{for all } n \in \mathbb{Z} \setminus \{0\}(\textit{resp. } n \in \mathbb{N}).$$

(ii) $\Rightarrow$ (iii): It is obvious.

(iii) $\Rightarrow$ (ii): Take $\epsilon > 0$ and $l > 0$. Then there exists $i_0 \in \mathbb{N}$ such that

$$l \in [l_{i_0 - 1},\ l_{i_0}).$$

Let L be $l_{i_0} - l$. Take $0 < \epsilon' < \frac{e^{-\omega L}\epsilon}{M}$. Then from the assumption, there exists $P > 0$ such that

$$\rho(l_{i_0} + nP) < \epsilon' \qquad \text{for all } n \in \mathbb{Z} \setminus \{0\} \text{ (resp. } n \in \mathbb{N}).$$

So we infer

$$\begin{aligned}
\rho(l + nP) &\leq Me^{\omega L}\rho(l + nP + L) \\
&= Me^{\omega L}\rho(l + nP + l_{i_0} - l) \\
&< Me^{\omega L}\rho(l_{i_0} + nP) \\
&< Me^{\omega L}\epsilon' \\
&< Me^{\omega L}\frac{e^{-\omega L}\epsilon}{M} \\
&= \epsilon.
\end{aligned}$$

(ii) $\Rightarrow$ (i): We obtain the conclusion by a similarly way to the proof of Theorem 2.1. □

Though the condition "$\lim_{\tau\to\infty} \rho(\tau) = 0$" is only a sufficient but not necessary condition for the translation semigroup to be chaotic in $C_{0,\rho}$, it is a necessary condition in L^p_ρ. So we obtain an equivalent condition to $\lim_{\tau\to\infty} \rho(\tau) = 0$.

Theorem 2.3. *Let I be $(-\infty, \infty)$ (resp. $I = [0, \infty)$), and let X be $C_{0,\rho}(I)$. Then for a translation semigroup $\{T(t)\}$, the following conditions are equivalent:*

(i) $\lim_{\tau\to\pm\infty} \rho(\tau) = 0$ *(resp. $\tau \to \infty$);*

(ii) *$\{T(t)\}$ is chaotic. In addition, for all $\epsilon > 0$ and for all $x \in X$ there exists t_0, for all $t \geq t_0$ there exists $v_t \in X_p$ such that*

$$\|x - v_t\| < \epsilon \quad \textit{and} \quad T(t)v_t = v_t.$$

The next theorem shows that when I is a half line, the denseness of the set of periodic points implies the hypercyclicity.

Theorem 2.4. *Let I be $[0,\infty)$ and X be $L^p_\rho(I)$ or $C_{0,\rho}(I)$. Then the set X_p of periodic points is dense in X if and only if $\{T(t)\}$ is chaotic.*

The next example is an application of Theorem 2.3.

Example 1. Let $C_0([0,\infty))$ be the space of continuous functions on $[0,\infty)$ which vanish at infinity. We shall consider the following partial differential equation on the space $C_0([0,\infty))$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - \omega u & \omega < 0 \\ u(0,x) = f(x) & f \in C_0^1([0,\infty)). \end{cases}$$

Then the solution is

$$u(t,x) = e^{-\omega t} f(x+t).$$

If we define an operator $\tilde{T}(t)$ on $C_0([0,\infty))$ by $\tilde{T}(t)f(x) = u(t,x)$, then $\{\tilde{T}(t)\}$ becomes a chaotic semigroup.

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