Some Duality Theorems of Set-Valued Optimization (Decision Theory in Mathematical Modelling)

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Some Duality Theorems of Set-Valued Optimization*

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Abstract

A set optimization problem with a set-valued objective function is investigated, and duality result is considered.

1 Introduction

Set-valued optimization has been investigated for about twenty years by many authors and various results concerned with the problem were obtained, see [1, 2, 3, 4, 6, 8, 9] and so on. Usually, this optimization is interpreted as a vector optimization problem with a set-valued objective function as follows:

\[(VP) \quad \text{Minimize} \quad F(x) \]
\[\text{subject to} \quad x \in S\]

where $S$ is a nonempty set, $(Z, \leq)$ is an ordered space, $F$ is a set-valued map from $S$ to $Z$, that is, $F : S \to 2^Z$. The aim of vector optimization problem (VP) is to find $x_0 \in S$, called solution, satisfying $F(x_0)$ includes a Pareto extremal point of $\bigcup_{x \in S} F(x)$, that is, there exists $z_0 \in F(x_0)$ such that if $z \in \bigcup_{x \in S} F(x)$ and $z \leq z_0$ then $z_0 = z$.

However, the aim of (VP) is not suitable for ‘set-valued optimization’ because such solutions are decided by one of the extremal elements of solution’s value. Recently, a set optimization problem with a set-valued objective function was introduced against vector optimization problem (VP), see [5]. Criteria of solutions of the optimization problem are obtained by comparisons of set-values of the objective function, these are called natural criteria. Our aim of this paper is to establish duality theory of such a set optimization problem with a set-valued objective function.

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The construction of this paper is the following: In Section 2, we mention some notations and definitions concerned with such a set optimization problem. In Section 3, we show an embedding theorem, and also we prove a strong duality theorem. In Section 4, we prove a saddle point theorem for our set optimization problem.

2 Set Optimization of Set-Valued Maps

Let $X$ be a nonempty set, $Y$ and $Z$ topological vector spaces, $K$ and $L$ solid, pointed, convex cones of $Y$, $Z$, respectively, $F$ and $G$ set-valued maps from $X$ to $Y$ and $Z$, respectively, that is $F : X \rightarrow 2^Z$, $G : X \rightarrow 2^Y$, and assume that $F(x) \neq \emptyset$ and $G(x) \neq \emptyset$ for each $x \in S$, and $S := \{x \in X \mid G(x) \cap (-K) \neq \emptyset\}$. Now we define problem $(SP)$ as follows:

$\text{(SP)}$ Minimize $F(x)$

subject to $x \in S$.

Before to define notions of solutions of problem $(SP)$, we mention about some set relations in ordered vector space $(Z, \leq_L)$. For $\emptyset \neq A, B \subseteq Z$,

$A \leq_L^{l} B \iff A + L \supset B$,

$A \leq_L^{u} B \iff A \subseteq B - L$.

In these notations, $l$ means lower and $u$ means upper: $A \leq_L^{l} B$ iff each element $b$ of $B$ has a lower bound in $A$, and $A \leq_L^{u} B$ iff each element $a$ of $A$ has an upper bound in $B$. We treat only relation $\leq_L^{l}$ in this paper.

The aim of problem $(SP)$ is to find the following solutions:

Definition 2.1 A vector $x_0 \in X$ is said to be

(i) a feasible solution of $(SP)$ if $x \in S$

(ii) a minimal solution of $(SP)$ if $x_0 \in S$, and if $x \in S$ and $F(x) \leq_L^{l} F(x_0)$ are satisfied, then $F(x_0) \leq_L^{l} F(x)$ is fulfilled.

If $F(x)$ is a singleton, that is $F(x)$ is written by $F(x) = \{f(x)\}$ for some map $f$ from $X$ to $Z$, these notions are equivalent to usual ones of 'set-valued optimization.'

3 Embedding Theorem and Duality Theorem

In the rest of paper, we assume that all values of set-valued map $F$ are nonempty compact convex. We denote $C(Z)$ as the family of all nonempty compact convex sets in $Z$.

First, we construct an ordered normed linear space $\mathcal{V}$ in which $C(Z)$ is embedded. On $C(Z)^2$ we define an equivalent relation $\sim$: for $(A, B), (C, D) \in C(Z)^2$,

$(A, B) \sim (C, D) \iff A + D + L = B + C + L.$
Let \([A, B]\) be the equivalence class includes \((A, B)\), and let \(\mathcal{V}\) be \(C(Z)^2/\sim\), the sets of all equivalence classes \([A, B]\). We define a vector structure on \(\mathcal{V}\) as follows: for \([A, B]\), \([C, D]\) \(\in \mathcal{V}\), sum and scalar product are defined by:

\[
[(A, B)] + [(C, D)] := [(A + C, B + D)]
\]

\[
\lambda \cdot [(A, B)] := \begin{cases} 
[(\lambda A, \lambda B)], & \lambda \geq 0 \\
[(-\lambda B, -\lambda A)], & \lambda < 0
\end{cases}
\]

Then we can show that \(\mathcal{V}\) is a vector space over the real field. Moreover, we define a norm \(\| \cdot \|\). For \([A, B]\) \(\in \mathcal{V}\),

\[
\|[(A, B)]\| := \inf \{\lambda \geq 0 | A + \lambda U \leq^L B, B + \lambda U \leq^L A\}
\]

then, \((\mathcal{V}, \| \cdot \|)\) is a normed space. Let \(\Pi := \{[(A, B)] \in \mathcal{V} | B \leq^L A\}\), then \(\Pi\) is a solid, pointed, convex cone in \(\mathcal{V}\), and we can derive a partial order \(\leq_{\Pi}\) in \(\mathcal{V}\):

\[
[(A, B)] \leq_{\Pi} [(C, D)] \iff [(C, D)] - [(A, B)] \in \Pi
\]

Finally, \(\mathcal{V}\) is an ordered normed space over the real field.

Now we show the following embedding theorem:

**Theorem 3.1** Let \(\varphi : C(Z) \rightarrow \mathcal{V}\) by

\[\varphi(A) := [(A, \{\theta\})], \ A \in C(Z)\]

then, the following are satisfied:

(i) For each \(A, B \in C(Z)\),

\[A \leq_{\Pi} B \iff \varphi(A) \leq_{\Pi} \varphi(B)\];

(ii) conditions a) and b) are equivalent:

a) \(x_0 \in S\) is a solution of set optimization (SP),

b) \(x_0 \in S\) is a solution of the following vector optimization (EP):

\[
\text{(EP)} \quad \begin{array}{l}
\text{Minimize} \quad \varphi(F(x)) \\
\text{subject to} \quad x \in S.
\end{array}
\]

From this result, we can use results of vector optimization with set-valued maps to solve set optimization with set-valued maps.

**Theorem 3.2** Let the following assumptions are satisfied:

(A1) \(F\) is nonempty compact convex values
(A2) $\forall x_1, x_2 \in X, \forall y_1 \in G(x_1), y_2 \in G(x_2), \forall \lambda \in (0,1), \exists (x, y) \in \text{Gr}(G)$ such that

$$
\begin{align*}
F(x) &\leq_L (1-\lambda)F(x_1) + \lambda F(x_2) \\
y &\leq_K (1-\lambda)y_1 + \lambda y_2
\end{align*}
$$

(A3) $\exists x' \in X$ such that $G(x') \cap (-\text{int}K) \neq \emptyset$

(A4) $x_0$ is a proper solution of set optimization of (SP)

then there exist $y^*_0 \in K^+ \setminus \{\theta\}$ and $\mu : \text{int}L \to (0, \infty)$ such that

- (i) $1/\mu$ is affine on int$L$
- (ii) for each $a \in \text{int}L$, $(T_a, \varphi(F(x_0)))$ is a weak maximizer of the weak dual problem of (EP),

where $T_a(y) = \langle y^*_0, y \rangle \mu(a)a$, $y \in Y$.

**Corollary 3.1** Under same assumption of the last theorem, there exist $y^*_0 \in K^+ \setminus \{\theta\}$ and $\mu : \text{int}L \to (0, \infty)$ with $1/\mu$ is affine on int$L$ such that

for any $a \in \text{int}L$, there does not exist $(x, y) \in \text{Gr}(G)$ such that

$$
F(x) + T_a(y) \leq_{\text{int}L} F(x_0)
$$

where $T_a(y) = \langle y^*_0, y \rangle \mu(a)a$, $y \in Y$.

## 4 Saddle Point Theorem

In this section, we consider a saddle point theorem of (SP). First, for primal problem (SP), we define dual problem (SD):

(SD) Maximize $\Phi(T)$

subject to $T \in \mathcal{M}$

where

- $\Phi(T) = \text{Min}(\varphi(L(X, T))|\Pi)$
- $L(x, T) = F(x) + T(G(x))$
- $\mathcal{M} = \{T \in \mathcal{L}(Y, Z)_+ \mid T = \langle y^*, \cdot \rangle a, y^* \in K^+ \setminus \{\theta\}, a \in \text{int}L\}$

**Definition 4.1** (Saddle Point) $(x_0, T_0) \in X \times \mathcal{M}$ is said to be a saddle point of $L$ if

$$
\varphi(L(x_0, T_0)) \cap \text{Max}(\varphi(L(x_0, \mathcal{M}))|\Pi) \cap \text{Min}(\varphi(L(X, T_0))|\Pi) \neq \emptyset.
$$

**Proposition 4.1** $(x_0, T_0) \in X \times \mathcal{M}$ is a saddle point of $L$ iff there exists $y_0 \in G(x_0)$ such that
(i) $F(x) + T_0(y) \leq^l F(x_0) + T_0(y_0), (x, y) \in \text{Gr}(G)$  
$\Rightarrow F(x_0) + T_0(y_0) \leq^l F(x) + T_0(y)$

(ii) $F(x_0) + T_0(y_0) \leq^l F(x_0) + T(y_0), T \in \mathcal{L}_+(Y, Z)$  
$\Rightarrow F(x_0) + T(y_0) \leq^l F(X_0) + T_0(y_0)$

**Theorem 4.1** (Saddle Point Theorem) If $(x_0, T_0)$ is a saddle point of $L$, then

1) $x_0$ is an optimal of (SP);

2) $T_0$ is an optimal of (SD);

3) $\varphi(F(x_0)) \cap \Phi(T_0) \cap \text{Max}(\Phi(M) | \Pi) \neq \emptyset$;

4) $G(x_0) \subset -K$;

5) $T_0(y) = \theta$ for all $y \in G(x_0)$.

Conversely, if 1) through 5) above and $F(x) = \text{Min}(F(x) | K)$ for each $x \in X$ hold, then $(x_0, T_0)$ is a saddle point of $L$.

**References**


