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Unbounded $C^*$-seminorms and $*$-representations of $*$-algebras

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1. INTRODUCTION

Unbounded $C^*$-seminorms on $*$-algebras in the sense that they are $C^*$-seminorms defined on $*$-subalgebras have appeared in many mathematical and physical subjects (for example, locally convex $*$-algebras and the quantum field theory etc.). But this systematical study has not yet done sufficiently. The main purpose of this paper is to do a systematical study of unbounded $C^*$-seminorms and to apply it to a study of unbounded $*$-representations.

The paper is organized as follows: In Section 2 we construct unbounded $*$-representations of a $*$-algebra from unbounded $C^*$-seminorms and investigate them. Let $A$ be a $*$-algebra. Let $p$ be a $C^*$-seminorm defined on $A$. Every $*$-representation of the Hausdorff completion of $(A, p)$ gives rise to a $*$-representation of $A$ into bounded Hilbert space operators. However, there are a number of situations in which natural $C^*$-seminorms are defined on $*$-subalgebras of $A$. Then they should lead to unbounded operator representations of $A$. An unbounded $m^*$-(resp. $C^*$-)seminorm is a submultiplicative $*$ (resp. $C^*$-) seminorm $p$ defined on a $*$-subalgebra $D(p)$ of $A$. Then $\mathfrak{M}_p \equiv \{ x \in D(p); ax \in D(p), \forall a \in A \}$ is a left ideal of $A$. It is shown that any $*$-representation $\Pi_p: A_p \longrightarrow \mathcal{B}(\mathcal{H})$ of the Hausdorff completion $A_p$ of $(D(p), p)$ leads to an unbounded $*$-representation $\pi_p$ of $A$ such that $\|\pi_p(x)\| \leq p(x)$ for all $x \in D(p)$. We denote by $\text{Rep}(A, p)$ the set of all such $*$-representations $\pi_p$ of $A$. In order to investigate representations in $\text{Rep}(A, p)$ in details, we introduce the notions of nondegenerate, finite, uniformly semifinite, semifinite and weakly semifinite unbounded $C^*$-seminorms, and show that if $p$ is (weakly) semifinite, then there exists a strongly nondegenerate $*$-representation $\pi_p$ in $\text{Rep}(A, p)$ such
that $\|\pi_p(x)\| = p(x)$ for all $x \in \mathcal{D}(p)$. Such a $\pi_p$ is called well-behaved. In Section 3 we consider the converse direction of Section 2. We construct an unbounded $C^*$-seminorm $r_\pi$ on $\mathcal{A}$ from a *-representation $\pi$ of $\mathcal{A}$ and a natural representation $\pi_{r_\pi}^N$ of $\mathcal{A}$ constructed from $r_\pi$ which is the restriction of the closure $\hat{\pi}$ of $\pi$. It is shown that $\pi$ is strongly nondegenerate if and only if $\pi_{r_\pi}^N$ is a well-behaved *-representation of $\mathcal{A}$. Further, it is shown that if $p$ is a weakly semifinite unbounded $C^*$-seminorm on $\mathcal{A}$ and $\pi_p$ is any well-behaved *-representation, then $r_{\pi_p}$ is a maximal extension of $p$.

2. REPRESENTATIONS INDUCED BY UNBOUNDED $C^*$-SEMINORMS

In this section we construct a family of *-representations of a *-algebra $\mathcal{A}$ induced by an unbounded $C^*$-seminorm on $\mathcal{A}$ and investigate the properties. We begin with the review of (unbounded) *-representations of $\mathcal{A}$. Throughout this section let $\mathcal{A}$ be a *-algebra. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ and let $\mathcal{L}^i(\mathcal{D})$ denote the set of all linear operators $X$ in $\mathcal{H}$ with the domain $\mathcal{D}$ for which $X\mathcal{D} \subset \mathcal{D}$, $\mathcal{D}(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^i(\mathcal{D})$ is a *-algebra under the usual operations and the involution $X \rightarrow X^* \equiv X^*|\mathcal{D}$. A *-subalgebra of the *-algebra $\mathcal{L}^i(\mathcal{D})$ is said to be an $O^*$-algebra on $\mathcal{D}$ in $\mathcal{H}$. A *-representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with a domain $\mathcal{D}$ is a *-homomorphism of $\mathcal{A}$ into $\mathcal{L}^i(\mathcal{D})$ and $\pi(1)=I$ if $\mathcal{A}$ has identity 1, and then we write $\mathcal{D}$ and $\mathcal{H}$ by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$, respectively. Let $\pi_1$ and $\pi_2$ be *-representations of $\mathcal{A}$. If $\mathcal{H}_{\pi_1}$ is a closed subspace of $\mathcal{H}_{\pi_2}$ and $\pi_1(x) \subset \pi_2(x)$ for each $x \in \mathcal{A}$, then $\pi_2$ is said to be an extension of $\pi_1$ and denoted by $\pi_1 \subset \pi_2$. In particular, if $\pi_1 \subset \pi_2$ and $\mathcal{H}_{\pi_1} = \mathcal{H}_{\pi_2}$, then $\pi_2$ is said to be an extension of $\pi_1$ as the same Hilbert space. Let $\pi$ be a *-representation of $\mathcal{A}$. If $\mathcal{D}(\pi)$ is complete with the graph topology $t_\pi$ defined by the family of seminorms $\{\|\bullet\|_{\pi(x)} \equiv \|\bullet\| + \|\pi(x)\bullet\|; x \in \mathcal{A}\}$, then $\pi$ is said to be closed. It is well known that $\pi$ is closed if and only if $\mathcal{D}(\pi) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\pi(x))$. The closure $\hat{\pi}$ of $\pi$ is defined by $\mathcal{D}(\hat{\pi}) = \bigcap_{x \in \mathcal{A}} \overline{\mathcal{D}(\pi(x))}$ and $\hat{\pi}(x)\xi = \overline{\pi(x)\xi}$ for $x \in \mathcal{A}$, $\xi \in \mathcal{D}(\hat{\pi})$. 


Then $\tilde{\pi}$ is the smallest closed extension of $\pi$. The weak commutant $\pi(A)'_w$ of $\pi$ is defined by

$$\pi(A)'_w = \{ C \in \mathcal{B}(H_\pi); C\pi(x)\xi = \pi(x^*)^*C\xi, \forall x \in A, \forall \xi \in \mathcal{D}(\pi) \},$$

where $\mathcal{B}(H_\pi)$ is the set of all bounded linear operators on $H_\pi$, and it is a weakly closed $*$-invariant subspace of $\mathcal{B}(H_\pi)$, but it is not necessarily an algebra. It is known that $\pi(A)'_w \mathcal{D}(\pi) \subset \mathcal{D}(\pi)$ if and only if $\pi(A)'_w$ is a von Neumann algebra and $\overline{\pi(x)}$ is affiliated with the von Neumann algebra $\left(\pi(A)'_w\right)$ for each $x \in A$.

**Definition 2.1.** A mapping $p$ of a subspace $\mathcal{D}(p)$ of $A$ into $\mathbb{R}^+=[0,\infty)$ is said to be an unbounded (semi) norm on $A$ if it is a (semi) norm on $\mathcal{D}(p)$, and $p$ is said to be an unbounded $m^*$-(resp. $C^*$-) (semi) norm on $A$ if $\mathcal{D}(p)$ is a $*$-subalgebra of $A$ and $p$ is a submultiplicative $*$-(resp. $C^*$-) (semi) norm on $\mathcal{D}(p)$.

If a seminorm $p$ on a $*$-algebra $A$ is a $C^*$-seminorm, that is, it satisfies the $C^*$-property $p(x^*x) = p(x)^2$, $\forall x \in A$, then it is a $m^*$-seminorm on $A$, that is, $p(x^*) = p(x)$ and $p(xy) \leq p(x)p(y)$ for $\forall x, y \in A$.

Let $p$ be an unbounded $C^*$-seminorm on $A$. We put

$$N_p = \{ x \in \mathcal{D}(p); p(x) = 0 \} \text{ and } \mathfrak{M}_p = \{ x \in \mathcal{D}(p); ax \in \mathcal{D}(p), \forall a \in A \}.$$

Then $N_p$ is a $*$-ideal of $\mathcal{D}(p)$ and $\mathfrak{M}_p$ is a left ideal of $A$, and the quotient $*$-algebra $\mathcal{D}(p)/N_p$ is a normed $*$-algebra with the $C^*$-norm $\|x + N_p\|_p = p(x)$ ($x \in \mathcal{D}(p)$). We denote by $A_p$ the $C^*$-algebra obtained by the completion of $\mathcal{D}(p)/N_p$, and denote by Rep$(A_p)$ the set of all $*$-representations $\Pi_p$ of the $C^*$-algebra $A_p$ on Hilbert space $H_{\Pi_p}$. Put

$$\text{FRep}(A_p) = \{ \Pi_p \in \text{Rep}(A_p); \Pi_p \text{ is faithful} \}$$

$$\text{FNRep}(A_p) = \{ \Pi_p \in \text{Rep}(A_p); \Pi_p \text{ is faithful and nondegenerate} \}.$$

It is well known that $\text{FNRep}(A_p) \neq \emptyset$. For each $\Pi_p \in \text{Rep}(A_p)$ we can define a bounded $*$-representation $\pi^0_p$ of $\mathcal{D}(p)$ on the Hilbert space $H_{\Pi_p}$ by

$$\pi^0_p(x) = \Pi_p(x + N_p), \quad x \in \mathcal{D}(p).$$

The natural question arises: Can we extend the bounded $*$-representation $\pi^0_p$ of the
*-algebra $\mathcal{D}(p)$ to a (generally unbounded) *-representation of the *-algebra $\mathcal{A}$? We show that this question has affirmative answer.

**Proposition 2.2.** Let $p$ be an unbounded $C^*$-seminorm on $\mathcal{A}$. For any $\Pi_p \in \text{Rep}(A_p)$, there exists a *-representation $\pi_p$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\pi_p}$ such that $\|\overline{\pi_p(b)}\| \leq p(b)$ for each $b \in \mathcal{D}(p)$. In particular, if $\Pi_p \in \text{FRep}(A_p)$, then $\|\overline{\pi_p(x)}\| = p(x)$ for each $x \in \mathfrak{R}_p$.

Proof. We put

$$\mathcal{D}(\pi_p) = \text{linear span of } \{\Pi_p(x + N_p)\xi; x \in \mathfrak{R}_p, \text{and } \xi \in \mathcal{H}_{\Pi_p}\}$$

$$\pi_p(a) \left( \sum_k \Pi_p(x_k + N_p)\xi_k \right) = \sum_k \Pi_p(ax_k + N_p)\xi_k \quad \text{(finite sums)}$$

for $a \in \mathcal{A}$, $\{x_k\} \subset \mathfrak{R}_p$ and $\{\xi_k\} \subset \mathcal{H}_{\Pi_p}$.

Since

$$\left( \Pi_p(ax + N_p)\xi \right) \left( \Pi_p(y + N_p)\eta \right) = \left( \xi \left( \Pi_p \left( (ax + N_p)^* (y + N_p) \right) \right) \right)$$

$$= \left( \xi \left( \Pi_p \left( x^* a^* y + N_p \right) \right) \eta \right)$$

$$= \left( \xi \left( \Pi_p \left( x^* + N_p \right) \Pi_p \left( a^* y + N_p \right) \right) \eta \right)$$

$$= \left( \Pi_p \left( x + N_p \right) \xi \right) \left( \Pi_p \left( a^* y + N_p \right) \eta \right)$$

for each $a \in \mathcal{A}$, $x, y \in \mathfrak{R}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$, it follows that $\pi_p(a)$ is a well-defined linear operator on $\mathcal{D}(\pi_p)$ for each $a \in \mathcal{A}$, so that it is easily shown that $\pi_p$ is a *-representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_{\pi_p} = \overline{\mathcal{D}(\pi_p)}$ (the closure of $\mathcal{D}(\pi_p)$ in $\mathcal{H}_{\Pi_p}$) with domain $\mathcal{D}(\pi_p)$. Take an arbitrary $b \in \mathcal{D}(p)$. By the definition of $\pi_p$ we have $\pi_p(b) = \pi_p^o(b)\mathcal{D}(\pi_p)$, and hence

$$\|\overline{\pi_p(b)}\| \leq \|\Pi_p(b + N_p)\| \leq \|b + N_p\| = p(b).$$

Suppose $\Pi_p \in \text{FRep}(A_p)$ and $x \in \mathfrak{R}_p$. It is sufficient to show that $\|\overline{\pi_p(x)}\| \geq p(x)$.

If $p(x) = 0$, then it is obvious. Suppose $p(x) \neq 0$. We put $y = \frac{x}{p(x)} \in \mathfrak{R}_p$. For each $\xi \in \mathcal{H}_{\Pi_p}$ with $\|\xi\| \leq 1$, we have

$$\|\Pi_p(y + N_p)\xi\| \leq \|\Pi_p(y + N_p)\|\|\xi\| = p(y)\|\xi\| \leq 1,$$

and so
Hence, we have \( \|\pi_p(x)\| \geq p(x) \). This completes the proof.

We have the following diagram:

\[
\begin{array}{cccc}
\mathcal{D}(p) & \xrightarrow{\text{completion}} & \mathcal{D}(p)/N_p & \xrightarrow{\Pi_p} A_p \ (C^*-\text{algebra}) \\
\pi_p^0 & \searrow & & \\
\Pi_p \circ (A_p) \ (C^*-\text{algebra on } \mathcal{H}_{\Pi_p}) & \downarrow & \pi_p (A) \ (O^*\text{-algebra in } \mathcal{H}_{\pi_p} \subset \mathcal{H}_{\Pi_p}).
\end{array}
\]

**Remark:** The *-representation \( \pi_p \) of \( A \) defined above by an unbounded \( C^* \)-seminorm \( p \) on \( A \) and an element \( \Pi_p \) of \( \text{Rep}(A_p) \) is non-zero if and only if \( A \mathfrak{N}_p \not\subset N_p \). In what follows, we discuss several situations keeping this in mind.

Let \( p \) be an unbounded \( C^* \)-seminorm on \( A \). We denote by \( \text{Rep}(A,p) \), \( \text{FRep}(A,p) \) and \( \text{FNRep}(A,p) \) the sets of all *-representations of \( A \) constructed as above by \( (A,p) \), that is,

\[
\begin{align*}
\text{Rep}(A,p) &= \{ \pi_p ; \Pi_p \in \text{Rep}(A_p) \}, \\
\text{FRep}(A,p) &= \{ \pi_p ; \Pi_p \in \text{FRep}(A_p) \}, \\
\text{FNRep}(A,p) &= \{ \pi_p ; \Pi_p \in \text{FNRep}(A_p) \}.
\end{align*}
\]
**Definition 2.3.** An unbounded $m^*$-seminorm $q$ on $A$ is said to be nondegenerate if $\mathcal{D}(q)^2$ is total in $\mathcal{D}(q)$ with respect to the seminorm $q$. An unbounded $m^*$-seminorm $q$ on $A$ is said to be finite if $\mathcal{D}(q)=\mathfrak{M}_q$; and $q$ is said to be uniformly semifinite if there exists a net $\{u_\alpha\}$ in $\mathfrak{M}_q$ such that $u_\alpha^*=u_\alpha$ and $q(u_\alpha)\leq 1$ for each $\alpha$ and $\lim q(xu_\alpha-x)=0$ for each $x \in \mathcal{D}(q)$; and $q$ is said to be semifinite if $\mathfrak{M}_q$ is dense in $\mathcal{D}(q)$ with respect to the seminorm $q$. An unbounded $C^*$-seminorm $p$ on $A$ is said to be weakly semifinite if $\text{FRep}^W(A, p) \equiv \{\pi_p \in \text{FRep}(A, p); \mathcal{H}_{\pi_p} = \mathcal{H}_{\pi_p}\} \neq \emptyset$. An element $\pi_p$ of $\text{Rep}^W(A, p)$ is said to be a well-behaved $*$-representation of $A$ in $\text{Rep}(A, p)$.

**Definition 2.4.** A $*$-representation $\pi$ of $A$ is said to be nondegenerate if $[\pi(A)\mathcal{D}(\pi)]=\mathcal{H}_\pi$; and $\pi$ is said to be strongly nondegenerate if there exists a left ideal $\mathcal{I}$ of $A$ contained in the bounded part $A_\mathcal{I} \equiv \{x \in A; \pi(x) \in \mathfrak{B}(\mathcal{H}_\pi)\}$ of $\pi$ such that $[\pi(\mathcal{I})\mathcal{H}_\pi]=\mathcal{H}_\pi$.

**Proposition 2.5.** Let $p$ be an unbounded $C^*$-seminorm on $A$. Then the following statements hold:

1. $\text{Rep}^W(A, p) \subset \text{FNRRep}(A, p)$ and every $\pi_p \in \text{Rep}^W(A, p)$ satisfies the following conditions (i), (ii) and (iii):
   
   (i) $[\pi_p(\mathfrak{M}_p)\mathcal{H}_{\pi_p}]=\mathcal{H}_{\pi_p}$, and $\pi_p$ is strongly nondegenerate.
   
   (ii) $\|\pi_p(x)\| = p(x), \forall x \in \mathcal{D}(p)$.
   
   (iii) $\pi_p(A)'' = \pi_p(\mathcal{D}(p))$ and $\pi_p(A)'' \subset \mathcal{D}(\pi_p)$.

   Conversely suppose $\pi_p \in \text{FRep}(A, p)$ satisfies conditions (i) and (ii) above. Then there exists an element $\pi_p^{WB}$ of $\text{Rep}^W(A, p)$ which is a representation of $\pi_p$.

2. Suppose $p$ is semifinite. Then $\text{Rep}^W(A, p) = \text{FNRRep}(A, p)$ and $\mathfrak{M}_p^2$ is total in $\mathcal{D}(p)$ with respect to $p$, and so $p$ is nondegenerate.

3. Suppose $p$ is uniformly semifinite. Then $A^* = A_\mathcal{I} \equiv \{a \in A; \exists k_a > 0 \text{ s.t. } p(ax) \leq k_a p(x), \forall x \in \mathfrak{M}_p\}$, $\|\pi_p(b)\| = \sup\{p(bx); x \in \mathfrak{M}_p \text{ and } p(x) \leq 1\}, \forall b \in A_\mathcal{I}$.
for each $\pi_p \in \text{FRep}(A, p)$.

(4) $p$ is finite if and only if $D(p)$ is a left ideal of $A$.

3. UNBOUNDED $C^*$-SEMINORMS DEFINED BY $*$-REPRESENTATIONS

In Section 2 we constructed a family $\text{Rep}(A, p)$ (resp. $\text{Rep}^\text{WB}(A, p)$) of $*$-representation of $A$ from an (resp. weakly semifinite) unbounded $C^*$-seminorm $p$ on $A$. Conversely we shall construct an unbounded $C^*$-seminorm $r_\pi$ on $A$ from a $*$-representation $\pi$ of $A$ and the natural representation $\pi^N_{r_\pi}$ of $A$ constructed from $r_\pi$, and investigate the relation $\pi$ and $\pi^N_{r_\pi}$. Let $\pi$ be a $*$-representation of $A$ on a Hilbert space $\mathcal{H}_\pi$. We put

$$A^\pi_b = \{x \in A; \overline{\pi(x)} \in \mathfrak{B}(\mathcal{H}_\pi)\} \text{ and } \pi_b(x) = \overline{\pi(x)}, \ x \in A^\pi_b.$$

Then $A^\pi_b$ is a $*$-subalgebra of $A$ and $\pi_b$ is a bounded $*$-representation of $A^\pi_b$ on $\mathcal{H}_\pi$. We denote by $C^*(\pi)$ the $C^*$-algebra generated by $\pi_b(A^\pi_b)$. We now define an unbounded $C^*$-seminorm $r_\pi$ on $A$ as follows;

$$D(r_\pi) = A^\pi_b \text{ and } r_\pi(x) = \|\pi_b(x)\|, \ x \in D(r_\pi).$$

Then we put

$$\Pi(x + N_{r_\pi}) = \pi_b(x), \ x \in A^\pi_b.$$

Since $\|\Pi(x + N_{r_\pi})\| = r_\pi(x) = \|x + N_{r_\pi}\|$, for each $x \in A^\pi_b$, it follows that $\Pi$ can be extended to a faithful $*$-representation $\Pi^N_{r_\pi}$ of $A_{r_\pi}$ on the Hilbert space $\mathcal{H}_\pi$. The $*$-representation $\pi^N_{r_\pi}$ of $A$ defined by $\Pi^N_{r_\pi}$ as above is called the natural representation of $A$ induced by $\pi$. Since $\mathcal{H}_{\Pi^N_{r_\pi}} = \mathcal{H}_\pi$, it follows that $\mathcal{H}_{\Pi^N_{r_\pi}}$ is a closed subspace of $\mathcal{H}_\pi$. We simply note the above method of the construction of $\pi^N_{r_\pi}$ by the following diagram:
We have the following results for the relation of $\pi$ and $\pi^N_{r\pi}$:

**Proposition 3.1.** Suppose $\pi$ is a $*$-representation of $A$ on a Hilbert space $\mathcal{H}_{\pi}$. Then the following statements hold:

1. $\pi^N_{r\pi} \subseteq \bar{\pi}$.
2. Suppose $\pi_b$ is nondegenerate. Then $\pi^N_{r\pi} \in \text{FNRep}(A, r\pi)$.
3. $\pi$ is strongly nondegenerate if and only if $\pi^N_{r\pi} \in \text{Rep}^{WB}(A, r\pi)$. If this is true, then $\pi^N_{r\pi}$ is strongly nondegenerate with $A^\pi_{r\pi} = A^\pi_b$, and $r\pi$ is weakly semifinite.
4. Suppose there exists a net $\{u_\alpha\}$ in $\mathcal{M}_{r\pi}$ such that $s-\lim_{\alpha} \pi(u_\alpha) = I$ and $s-\lim_{\alpha} \pi(a u_\alpha) = \pi(a)$ for each $a \in A$. Then $\pi^N_{r\pi} = \bar{\pi}$.

By Proposition 3.1 we have the following diagram:
We here investigate the relations of unbounded $C^*$-seminorms $p$ and $r_\pi$ and the $*$-representation $\pi_p$ and $\pi_{r_\pi}$. We first define an order relation among unbounded seminorms as follows:

**Definition 3.2.** Let $p$ and $q$ be unbounded seminorms on $A$. We say that $p$ is an extension of $q$ (or $q$ is a restriction of $p$) if $\mathcal{D}(q) \subset \mathcal{D}(p)$ and $q(x) = p(x)$ for each $x \in \mathcal{D}(q)$, and then denote by $q \subset p$.

We denote by $C^*N(A)$ the set of all unbounded $C^*$-seminorms on $A$. Then $C^*N(A)$ is an ordered set with the order $\subset$. For any $p \in C^*N(A)$ we put

$$C^*N(p) = \{ q \in C^*N(A) ; p \subset q \}.$$  

Then it follows from Zorn's lemma that $C^*N(p)$ has a maximal element. We show that if $p$ is weakly semifinite then $r_\pi$ is a maximal element of $C^*N(p)$.

**Lemma 3.3.** Let $p$ and $r$ be unbounded $C^*$-seminorms on $A$. Suppose $p \subset r$. Then, for any $\pi_p \in \text{Rep}(A,p)$ there exists an element $\pi_r$ of $\text{Rep}(A,r)$ such that $\pi_p \subset \pi_r$.

**Proposition 3.4.** Suppose $p$ is a weakly semifinite unbounded $C^*$-seminorm on $A$ and $\pi_p \in \text{Rep}^\text{WB}(A,p)$. Then $r_\pi$ is a maximal element of $C^*N(p)$ and $r_\pi = r_{\pi'}$ for each $\pi_p, \pi' \in \text{Rep}^\text{WB}(A,p)$.

By Proposition 3.1, (3) and Proposition 3.4 we have the following

**Corollary 3.5.** Suppose $\pi$ is a strongly nondegenerate $*$-representation of $A$. Then $r_\pi$ is maximal.

For the relation of $*$-representation $\pi_p$ and $\pi_{r_\pi}^N$ we have the following

**Proposition 3.6.** Suppose $p$ is a weakly semifinite unbounded $C^*$-seminorm on $A$ and $\pi_p \in \text{Rep}^\text{WB}(A,p)$. Then $\pi_p \subset \pi_{r_\pi}^N$ and $\pi_{r_\pi}^N = \pi_p$.