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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1998), 1072: 85-90</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-12</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62582">http://hdl.handle.net/2433/62582</a></td>
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<td>Type</td>
<td>Departmental Bulletin Paper</td>
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UNIFORMLY SHADOWING PROPERTY

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Let $M$ be a closed $C^\infty$ manifold and $C^r(M)$ be the set of all $C^r$-differentiable maps endowed with the $C^r$-topology ($r \geq 1$). $D_x f$ is the derivative of $f$ at $x$. Denote as $\tilde{M}$ the topological product space $\prod_{-\infty}^{\infty} M$ and define a compatible metric $\tilde{d}$ on $\tilde{M}$ by $\tilde{d}((x_n), (y_n)) = \sum_{-\infty}^{\infty} d(x_n, y_n)/2^{|n|}$ for $(x_n), (y_n) \in \tilde{M}$, where $d$ is a metric on $M$ induced by a Riemannian metric. We define a continuous map $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$ by

$$\tilde{f}((x_n)) = (f(x_n)).$$

Then the projection $P^0 : \tilde{M} \rightarrow M$ defined by $P^0((x_n)) = x_0$ satisfies $P^0 \circ \tilde{f} = f \circ P^0$. For a subset $\Lambda$ an $\tilde{f}$-invariant set $\Lambda_f$ is defined by

$$\Lambda_f = \{(x_n) \in \tilde{M} : x_n \in \Lambda, f(x_n) = x_{n+1}, n \in \mathbb{Z}\}.$$

If $\Lambda_f \neq \emptyset$ then $\tilde{f}|_{\Lambda_f} : \Lambda_f \rightarrow \Lambda_f$ is a surjective homeomorphism. Remark that $\Lambda_f = M_f \neq \emptyset$ when $\Lambda = M$. We say that each element of $M_f$ is an orbit of $f$. 
For $\delta \geq 0$ a sequence $\{x^i\}_{i \in \mathbb{Z}} \subset M$ is called a $\delta$-pseudo-orbit of $f$ if $d(f(x^i), x^{i+1}) \leq \delta$ for every $i \in \mathbb{Z}$. A sequence $\{x^i\}_{i \in \mathbb{Z}} \subset M$ is said to be $\varepsilon$-traced by an orbit $(y_i) \in M_f$ if $d(x^i, y_i) < \varepsilon$ for every $i \in \mathbb{Z}$. We say that $f$ has the shadowing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudo-orbit of $f$ can be $\varepsilon$-traced by an orbit of $f$.

A sequence $\{\tilde{x}^i\}_{i \in \mathbb{Z}} \subset M_f$ is called an orbit of $\tilde{f}$ if $\tilde{f}(\tilde{x}^i) = \tilde{x}^{i+1}$. For $\delta \geq 0$ a sequence $\{\tilde{x}^i\}_{i \in \mathbb{Z}} \subset M_f$ is a $\delta$-pseudo-orbit of $\tilde{f}$ if $\tilde{d}(\tilde{f}(\tilde{x}^i), \tilde{x}^{i+1}) \leq \delta$ for every $i \in \mathbb{Z}$. A sequence $\{\tilde{x}^i\}_{i \in \mathbb{Z}} \subset \tilde{M}$ is said to be $\varepsilon$-traced by an orbit $(\tilde{y}^i)$ of $\tilde{f}$ if $\tilde{d}(\tilde{x}^i, \tilde{y}^i) < \varepsilon$ for every $i \in \mathbb{Z}$. We say that $\tilde{f}$ satisfies the shadowing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudo-orbit of $\tilde{f}$ can be $\varepsilon$-traced by an orbit of $\tilde{f}$.

We say that $\tilde{f}$ satisfies $C^r$ uniformly shadowing property if there is a neighborhood $\mathcal{U}(f)$ of $f$ in $C^r(M)$ with the property that for $\varepsilon > 0$ there is $\delta > 0$ such that for $g \in \mathcal{U}(f)$ every $\delta$-pseudo-orbit of $\tilde{g}$ is $\varepsilon$-traced by an orbit of $\tilde{g}$.

Let $\pi : TM \to M$ be a tangent bundle of $M$ and $\| \cdot \|$ be a Riemannian metric on $TM$. Define a subset of the product topological space $\tilde{M} \times TM$ by

$$T\tilde{M} = \{ (\tilde{x}, v) \in \tilde{M} \times TM : P^0(\tilde{x}) = \pi(v) \}$$
and define a Finsler $|| \cdot ||$ on $T\tilde{M}$ by $||(\tilde{x}, v)|| = ||v||$. Then $\tilde{\pi}: T\tilde{M} \to \tilde{M}$ defined by $\tilde{\pi}(\tilde{x}, v) = \tilde{x}$ is a $C^0$-vector bundle over $\tilde{M}$. Define the projection $\tilde{P}^0: T\tilde{M} \to TM$ by $\tilde{P}^0(\tilde{x}, v) = v$. Then,

$$\tilde{P}^0|T_{\tilde{x}}\tilde{M} : T_{\tilde{x}}\tilde{M} \to T_{P^0(\tilde{x})}M$$

is a linear isomorphism where $T_{\tilde{x}}\tilde{M} = \tilde{\pi}^{-1}(\tilde{x})$. A linear bundle map $D\tilde{f} : T\tilde{M} \to T\tilde{M}$ covering $\tilde{f}$ is defined by

$$D\tilde{f}(\tilde{x}, v) = (\tilde{f}(\tilde{x}), D_{P^0(\tilde{x})}f(v)).$$

Then we have $D\tilde{f}(T_{\tilde{x}}\tilde{M}) \subset T_{\tilde{f}(\tilde{x})}\tilde{M}$ and $\tilde{P}^0 \circ D\tilde{f} = Df \circ \tilde{P}^0$. To simplify the notation we write $D_{\tilde{x}}\tilde{f} = D\tilde{f}|T_{\tilde{x}}\tilde{M}$. For a subset $\tilde{\Lambda}$ define

$$T\tilde{M}|\tilde{\Lambda} = \bigcup_{\tilde{x} \in \tilde{\Lambda}} T_{\tilde{x}}\tilde{M}.$$ 

A closed $f$-invariant set $\Lambda (f(\Lambda) = \Lambda)$ is said to be hyperbolic if $T\tilde{M}|\Lambda_f$ splits into the Whitney sum $T\tilde{M}|\Lambda_f = E^s \oplus E^u$ of subbundles $E^s$ and $E^u$, and there are $C > 0$ and $0 < \lambda < 1$ such that

(i) $D\tilde{f}(E^s) \subset E^s$ and $D\tilde{f}(E^u) = E^u$,

(ii) $D\tilde{f}|E^u : E^u \to E^u$ is invertible,

(iii) $||D\tilde{f}^n|E^s|| \leq C\lambda^n$ and $||(D\tilde{f}|E^u)^{-n}|| \leq C\lambda^n$ for $n \geq 0$,

where $|| T ||$ denotes the supremum norm of a linear bundle map $T$. The number $\lambda$ is called the skewness of the hyperbolic set $\Lambda$. For $\varepsilon > 0$ and
\( \tilde{x} \in M_f \) the local stable and the local unstable manifolds are defined by

\[
W^s_\epsilon(\tilde{x}, f) = \{y \in M : d(x_n, f^n(y)) \leq \epsilon \text{ for } n \geq 0\},
\]
\[
W^u_\epsilon(\tilde{x}, f) = \left\{ y \in M \begin{array}{l}
\text{there exists } \tilde{y} \in M_f \text{ such that } y_0 = y \\
\text{and } d(x_{-n}, y_{-n}) \leq \epsilon \text{ for } n \geq 0
\end{array} \right\}.
\]

Then, \( W^s_\epsilon(\tilde{x}, f) = W^s_\epsilon(\tilde{y}, f) \) for \( \tilde{x}, \tilde{y} \in M_f \) with \( x_0 = y_0 \).

For \( \tilde{x} \in M_f \) the stable and the unstable sets are defined by

\[
W^s(\tilde{x}, f) = \{y \in M : \lim_{n \to \infty} d(x_n, f^n(y)) = 0\},
\]
\[
W^u(\tilde{x}, f) = \left\{ y \in M \begin{array}{l}
\text{there is } \tilde{y} \in M_f \text{ satisfying } y_0 = y \\
\text{and } \lim_{n \to \infty} d(x_{-n}, y_{-n}) = 0
\end{array} \right\}.
\]

Then, \( W^s(\tilde{x}, f) = W^s(\tilde{y}, f) \) for \( \tilde{x}, \tilde{y} \in M_f \) with \( x_0 = y_0 \). If \( \Lambda \) is a hyperbolic set, for \( \tilde{x} \in \Lambda_f \) we have

\[
W^s(\tilde{x}, f) = \bigcup_{n=0}^{\infty} f^{-n}(W^s_\epsilon(f^n(\tilde{x}), f)), \quad W^u(\tilde{x}, f) = \bigcup_{n=0}^{\infty} f^n(W^u_\epsilon(f^{-n}(\tilde{x}), f)).
\]

Remark that \( W^\sigma(\tilde{x}, f) \) (\( \sigma = s, u \)) are not always the manifolds like the stable and unstable manifolds given by diffeomorphisms. However we can define the transversality condition between \( W^s(\tilde{x}, f) \) and \( W^u(\tilde{y}, f) \) as follows.

Let \( \tilde{y} \) and \( \tilde{z} \) be points in \( \Lambda_f \). We say that \( W^s(\tilde{y}, f) \) is transversal to \( W^u(\tilde{z}, f) \) if \( f^{n+m} | W^u_\epsilon(\tilde{f}^{-m}(\tilde{z}), f) \) is transversal to \( W^s_\epsilon(\tilde{f}^n(\tilde{y}), f) \) for \( \epsilon > 0 \) small enough and \( n, m \geq 0 \).
The non-wandering set $\Omega(f)$ is defined by

$$\Omega(f) = \left\{ x \in M \middle| \text{for any neighborhood } U \text{ of } x \text{ there is } n > 0 \right\}.$$  

Obviously, $\Omega(f)$ is closed and satisfies that $f(\Omega(f)) \subset \Omega(f)$ and $Per(f) \subset \Omega(f)$, where $Per(f)$ denotes the set of all periodic points of $f$. A differentiable map $f$ is said to satisfy Axiom $A$ if

(i) $Per(f)$ is dense in $\Omega(f)$,

(ii) $\Omega(f)$ is hyperbolic.

We say that an Axiom A differentiable map $f$ satisfies the strong transversality if $W^s(\tilde{y}, f)$ is transversal to $W^u(\tilde{z}, f)$ for $\tilde{y}, \tilde{z} \in \Omega(f)_f$.

**Theorem.** If $C^1$-differentiable map $f$ satisfies both Axiom $A$ and the strong transversality, then $\tilde{f}$ satisfies $C^1$ uniformly shadowing property.

This result was proved by Sakai for the class of $C^1$-diffeomorphisms. The full proof of our theorem will appear elsewhere.

**REFERENCES**


