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APPLICATIONS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $U=\{z\in C:|z|<1\}$. A function $f(z)$ in $\mathcal{A}$ is said to be starlike in U if it satisfies

(1.2) $$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0\quad(z\in U).$$

We denote by $\mathcal{S}^*$ the subclass of $\mathcal{A}$ consisting of all starlike functions in U. Further, a function $f(z)$ belonging to $\mathcal{A}$ is said to be convex in U if it satisfies

(1.3) $$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0\quad(z\in U).$$

Also, we denote by $\mathcal{K}$ the subclass of $\mathcal{A}$ consisting of all such functions.

It is well-known that

(1.4) $$f(z)\in\mathcal{S}^*\Longrightarrow|a_n|\leq n\quad(n\geq 2)$$

and

(1.5) $$f(z)\in\mathcal{K}\Longrightarrow|a_n|\leq 1\quad(n\geq 2)\quad(\text{cf.}[4]).$$

A function $f(z)\in\mathcal{A}$ is called to be uniformly starlike in U if it satisfies

(1.6) $$\operatorname{Re}\left\{\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\right\}\geq 0$$

for all $(z,\zeta) \in U \times U$. The subclass of $\mathcal{A}$ consisting of all functions which are uniformly starlike in U is denoted by UST. Furthermore, a function $f(z) \in \mathcal{A}$ is said to be uniformly convex in U if it satisfies

$$\text{Re}\left\{1+(z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geq 0 \tag{1.7}$$

for all $(z,\zeta) \in U \times U$. We also denote by UCV the subclass of $\mathcal{A}$ consisting of all uniformly convex functions in U. The classes UST and UCV were introduced by Goodman ([1], [2]).

The generalized hypergeometric series ${}_pF_q(z)$ defined by

$$ {}_pF_q(z) \equiv {}_pF_q\left(\begin{matrix} a_1, a_2, \cdots, a_p \\ \\ b_1, b_2, \cdots, b_q \end{matrix}\; ; z\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{i=1}^{q}(b_i)_n} \frac{z^n}{(1)_n} \quad (p \leq q+1), \tag{1.8}$$

where p and q are non-negative integers, a_j $(j = 1, 2, \cdots, p)$ and b_i $(i = 1, 2, \cdots, q)$ are complex numbers with $b_i \neq 0, -1, -2, \cdots$. Here $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in N). \end{cases} \tag{1.9}$$

If we set

$$\omega = \sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j, \tag{1.10}$$

we see that the series ${}_pF_q(z)$, with $p = q+1$, is

(i) absolutely convergent for $|z| = 1$ if Re $(\omega) > 0$,

(ii) conditionally convergent for $|z| = 1$, $z \neq 1$ if $-1 <$ Re $(\omega) \leq 0$, *and*

(iii) divergent for $|z| = 1$ if Re $(\omega) \leq -1$.

If $p < q+1$ and Re $(\omega) > 0$, then the ${}_pF_q(z)$ series (1.8) is absolutely convergent for $|z| < \infty$ (cf. Srivastava [6]).

2. Uniformly starlikeness

We discuss the uniformly starlikeness of the generalized hypergeometric functions with some conditions.

Lemma 2.1 **([3])** *If a function $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}(3n-2)|a_n| \leq 1, \tag{2.1}$$

then $f(z) \in UST$.

Now, we derive

Theorem 2. 1 *Let* $f(z) \in \mathcal{S}^*$, $\text{Re } b_i > 0$ $(i = 1, 2, \cdots, q)$, *and* $\sum_{i=1}^{q} \text{Re } b_i > \sum_{j=1}^{p} |a_j| + 2$. *If*

$$(2.2) \qquad 3\ {}_{p+2}F_{q+2}\left(\begin{array}{c} |a_1|, |a_2|, \cdots, |a_p|, 2, 2 \\ \\ \text{Re } b_1, \text{Re } b_2, \cdots, \text{Re } b_q, 1, 1 \end{array} ; 1\right) - 2\ {}_{p+1}F_{q+1}\left(\begin{array}{c} |a_1|, |a_2|, \cdots, |a_p|, 2 \\ \\ \text{Re } b_1, \text{Re } b_2, \cdots, \text{Re } b_q, 1 \end{array} ; 1\right) \leq 2,$$

then $(f * z\ {}_pF_q)(z) \in UST$, *where* $(s * t)(z)$ *denotes the Hadamard product of* $s(z)$ *and* $t(z)$.

Proof. By virtue of Lemma 2.1, if

$$(2.3) \qquad \sum_{n=2}^{\infty}(3n-2)\left|\frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_n\right| \leq 1$$

is satisfied, then $(f * z\ {}_pF_q)(z) \in UST$. Indeed, we have

$$(2.4) \qquad \begin{aligned} &\sum_{n=2}^{\infty}(3n-2)\left|\frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_n\right| \\ &\leq 3\sum_{n=0}^{\infty}\left(\frac{\prod_{j=1}^{p}(|a_j|)_{n+1}}{\prod_{i=1}^{q}(\text{Re } b_i)_{n+1}}\frac{(n+2)^2}{(1)_{n+1}}\right) - 2\sum_{n=2}^{\infty}\left(\frac{\prod_{j=1}^{p}(|a_j|)_{n+1}}{\prod_{i=1}^{q}(\text{Re } b_i)_{n+1}}\frac{n+2}{(1)_{n+1}}\right) \\ &= 3\sum_{n=0}^{\infty}\left(\frac{\prod_{j=1}^{p}(|a_j|)_n}{\prod_{i=1}^{q}(\text{Re } b_i)_n}\frac{((2)_n)^2}{((1)_n)^3}\right) - 2\sum_{n=0}^{\infty}\left(\frac{\prod_{j=1}^{p}(|a_j|)_n}{\prod_{i=1}^{q}(\text{Re } b_i)_n}\frac{(2)_n}{((1)_n)^2}\right) - 1 \\ &= 3\ {}_{p+2}F_{q+2}\left(\begin{array}{c} |a_1|, |a_2|, \cdots, |a_p|, 2, 2 \\ \\ \text{Re } b_1, \text{Re } b_2, \cdots, \text{Re } b_q, 1, 1 \end{array} ; 1\right) \\ &\quad -2\ {}_{p+1}F_{q+1}\left(\begin{array}{c} |a_1|, |a_2|, \cdots, |a_p|, 2 \\ \\ \text{Re } b_1, \text{Re } b_2, \cdots, \text{Re } b_q, 1 \end{array} ; 1\right) - 1, \end{aligned}$$

because $|a_n| \leq n \quad (n \geq 2)$ for $f(z) \in \mathcal{S}^*$. This completes the proof of Theorem 2.1.

Applying the Gauss summation theorem given by

$$(2.5) \qquad {}_2F_1\left(\begin{array}{c} a_1, a_2 \\ \\ b_1 \end{array} ; 1\right) = \frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)}$$

with $\text{Re } (b_1 - a_1 - a_2) > 0$, we have

Corollary 2. 1 *Let* $f(z) \in \mathcal{S}^*$, $\mathrm{Re}\ b_i > 0$ $(i = 1, 2, \cdots, q)$, $\mathrm{Re}\ b_m > |a_m| + 1$ $(m = 1, 2, \cdots, q-3)$, $\mathrm{Re}\ b_{q-2} > |a_{q-2}| + 2$, *and* $\mathrm{Re}\ b_q > |a_q| + |a_{q+1}|$. *If*

$$(2.6)\qquad \frac{(\mathrm{Re}\ b_{q-2} - 2)\Gamma(\mathrm{Re}\ b_q - 1)\Gamma(\mathrm{Re}\ b_q - |a_q| - |a_{q+1}|)}{(\mathrm{Re}\ b_{q-1} - |a_{q-1}| - 2)\Gamma(\mathrm{Re}\ b_q - |a_q| - 1)\Gamma(\mathrm{Re}\ b_q - |a_{q+1}|)} \times \left\{ \prod_{m=1}^{q} \left(\frac{\mathrm{Re}\ b_m - 1}{\mathrm{Re}\ b_m - |a_m| - 1} \right) \right\} \left(3\frac{\mathrm{Re}\ b_{q-2} - 2}{\mathrm{Re}\ b_{q-2} - |a_{q-2}| - 2} - 2 \right) \leq 2$$

is satisfied, then $(f * z\ {}_{q+1}F_q)(z) \in UST$.

Similarly, we have

Theorem 2. 2 *Let* $f(z) \in \mathcal{K}$, $\mathrm{Re}\ b_i > 0$ $(i = 1, 2, \cdots, q)$, *and* $\sum_{i=1}^{q} \mathrm{Re}\ b_i > \sum_{j=1}^{p} |a_j| + 1$. *If*

$$(2.7)\qquad 3\ {}_{p+1}F_{q+1}\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p|, 2 \\ \mathrm{Re}\ b_1, \mathrm{Re}\ b_2, \cdots, \mathrm{Re}\ b_q, 1 \end{matrix} ; 1 \right) - 2\ {}_pF_q\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p| \\ \mathrm{Re}\ b_1, \mathrm{Re}\ b_2, \cdots, \mathrm{Re}\ b_q \end{matrix} ; 1 \right) \leq 2,$$

is satisfied, then $(f * z\ {}_pF_q)(z) \in UST$.

Corollary 2. 2 *Let* $f(z) \in \mathcal{K}$, $\mathrm{Re}\ b_i > 0$ $(i = 1, 2, \cdots, q)$, $\mathrm{Re}\ b_m > |a_m| + 1$ $(m = 1, 2, \cdots, q-2)$, $\mathrm{Re}\ b_{q-1} > |a_{q-1}| + 2$, *and* $\mathrm{Re}\ b_q > |a_q| + |a_{q+1}|$. *If*

$$(2.8)\qquad \frac{\Gamma(\mathrm{Re}\ b_q - 1)\Gamma(\mathrm{Re}\ b_q - |a_q| - |a_{q+1}|)}{\Gamma(\mathrm{Re}\ b_q - |a_q| - 1)\Gamma(\mathrm{Re}\ b_q - |a_{q+1}|)} \times \left\{ \prod_{m=1}^{q} \left(\frac{\mathrm{Re}\ b_m - 1}{\mathrm{Re}\ b_m - |a_m| - 1} \right) \right\} \left(3\frac{\mathrm{Re}\ b_{q-1} - 2}{\mathrm{Re}\ b_{q-1} - |a_{q-1}| - 2} - 2 \right) \leq 2$$

is satisfied, then $(f * z\ {}_{q+1}F_q)(z) \in UST$.

If we take $f(z) = \frac{z}{1-z} \in \mathcal{K}$ in Theorem 2.2, then we obtain

Corollary 2. 3 *Let* $\mathrm{Re}\ b_i > 0$ $(i = 1, 2, \cdots, q)$, *and* $\sum_{i=1}^{q} \mathrm{Re}\ b_i > \sum_{j=1}^{p} |a_j| + 1$. *If*

$$(2.9)\qquad 3\ {}_{p+1}F_{q+1}\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p|, 2 \\ \mathrm{Re}\ b_1, \mathrm{Re}\ b_2, \cdots, \mathrm{Re}\ b_q, 1 \end{matrix} ; 1 \right) - 2\ {}_pF_q\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p| \\ \mathrm{Re}\ b_1, \mathrm{Re}\ b_2, \cdots, \mathrm{Re}\ b_q \end{matrix} ; 1 \right) \leq 2,$$

is satisfied, then $z\ {}_pF_q(z) \in UST$.

3. Uniformly convexity

For uniformly convex functions Rϕnning [5] proved that $f(z) \in UCV$ if and only if $f(z) \in \mathcal{A}$ satisfies

$$\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U). \tag{3.1}$$

From this fact, we have

Lemma 3. 1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n^2|a_n| \leq 1, \tag{3.2}$$

then $f(z) \in UCV$.

With the above lemma, we get

Theorem 3. 1 *Let $f(z) \in \mathcal{S}^*$, $\text{Re } b_i > 0$ $(i = 1, 2, \cdots, q)$, and $\sum_{i=1}^{q} \text{Re } b_i > \sum_{j=1}^{p} |a_j| + 3$. If*

$$ {}_{p+3}F_{q+3}\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p|, 2, 2, 2 \\ \\ \text{Re } b_1, \text{Re } b_2, \cdots, \text{Re } b_q, 1, 1, 1 \end{matrix} ; 1\right) \leq 2 \tag{3.3}$$

is satisfied, then $(f * z\ {}_pF_q)(z) \in UCV$.

Corollary 3. 1 *Let $f(z) \in \mathcal{S}^*$, $\text{Re } b_i > 0$ $(i = 1, 2, \cdots, q)$, $\text{Re } b_m > |a_m| + 1$ $(m = 1, 2, \cdots, q-4)$, $\text{Re } b_{q-3} > |a_{q-3}| + 2$, and $\text{Re } b_{q-2} > |a_{q-2}| + 2$, $\text{Re } b_{q-1} > |a_{q-1}| + 2$, and $\text{Re } b_q > |a_q| + |a_{q+1}|$. If*

$$\frac{(\text{Re } b_{q-3} - 2)(\text{Re } b_{q-2} - 2)(\text{Re } b_{q-1} - 2)}{(\text{Re } b_{q-3} - |a_{q-3}| - 2)(\text{Re } b_{q-2} - |a_{q-2}| - 2)(\text{Re } b_{q-1} - |a_{q-1}| - 2)} \times \frac{\Gamma(\text{Re } b_q - 1)\Gamma(\text{Re } b_q - |a_q| - |a_{q+1}|)}{\Gamma(\text{Re } b_q - |a_q| - 1)\Gamma(\text{Re } b_q - |a_{q+1}|)} \left\{\prod_{m=1}^{q}\left(\frac{\text{Re } b_m - 1}{\text{Re } b_m - |a_m| - 1}\right)\right\} \leq 2 \tag{3.4}$$

is satisfied, then $(f * z\ {}_{q+1}F_q)(z) \in UCV$.

Theorem 3. 2 *Let $f(z) \in \mathcal{K}$, $\text{Re } b_i > 0$ $(i = 1, 2, \cdots, q)$, and $\sum_{i=1}^{q} \text{Re } b_i > \sum_{j=1}^{p} |a_j| + 2$. If*

$$ {}_{p+2}F_{q+2}\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p|, 2, 2 \\ \\ \text{Re } b_1, \text{Re } b_2, \cdots, \text{Re } b_q, 1, 1 \end{matrix} ; 1\right) \leq 2 \tag{3.5}$$

is satisfied, then $(f * z\ {}_pF_q)(z) \in UCV$.

Corollary 3. 2 *Let* $f(z) \in \mathcal{K}$, $\mathrm{Re}\, b_i > 0$ $(i = 1, 2, \cdots, q)$, $\mathrm{Re}\, b_m > |a_m| + 1$ $(m = 1, 2, \cdots, q-3)$, $\mathrm{Re}\, b_{q-2} > |a_{q-2}| + 2$, $\mathrm{Re}\, b_{q-1} > |a_{q-1}| + 2$, *and* $\mathrm{Re}\, b_q > |a_q| + |a_{q+1}|$. *If*

$$\frac{(\mathrm{Re}\, b_{q-2} - 2)(\mathrm{Re}\, b_{q-1} - 2)\Gamma(\mathrm{Re}\, b_q - 1)\Gamma(\mathrm{Re}\, b_q - |a_q| - |a_{q+1}|)}{\Gamma(\mathrm{Re}\, b_q - |a_q| - 1)\Gamma(\mathrm{Re}\, b_q - |a_{q+1}|)} \times \left\{ \prod_{m=1}^{q} \left(\frac{\mathrm{Re}\, b_m - 1}{\mathrm{Re}\, b_m - |a_m| - 1} \right) \right\} \leq 2 \tag{3.6}$$

is satisfied, then $(f * z\ {}_{q+1}F_q)(z) \in UCV$.

Corollary 3. 3 *Let* $\mathrm{Re}\, b_i > 0$ $(i = 1, 2, \cdots, q)$, *and* $\sum_{i=1}^{q} \mathrm{Re}\, b_i > \sum_{j=1}^{p} |a_j| + 2$. *If*

$${}_{p+2}F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p|, 2, 2 \\ \mathrm{Re}\, b_1, \mathrm{Re}\, b_2, \cdots, \mathrm{Re}\, b_q, 1, 1 \end{matrix} ; 1 \right) \leq 2 \tag{3.7}$$

is satisfied, then $z\ {}_pF_q(z) \in UCV$.

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