FOURIER-JACOBI TYPE SPHERICAL FUNCTIONS ON $Sp(2, \mathbb{R})$; 
THE CASE OF $P_J$-PRINCIPAL SERIES AND DISCRETE SERIES

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1. Introduction
In this note, we study a kind of generalized Whittaker models, or equally, of generalized spherical functions associated with automorphic forms on the real symplectic group of degree two. We call these spherical functions 'Fourier-Jacobi type', since these are closely connected with the coefficients of the 'Fourier-Jacobi expansions' of (holomorphic or non-holomorphic) automorphic forms. Also these can be considered as a non-holomorphic analogue of the local Whittaker-Shintani functions on $Sp(2, \mathbb{R})$ of Fourier-Jacobi type in the paper of Murase and Sugano [6].

2. Preliminaries

2.1. Groups and algebras. We denote by $\mathbb{Z}_{\geq m}$ the set of integers $n$ such that $n \geq m$. Moreover, we use the convention that unwritten components of a matrix are zero.

Let $G$ be the real symplectic group $Sp(2, \mathbb{R})$ of degree two given by

$$Sp(2, \mathbb{R}) = \left\{ g \in M_4(\mathbb{R}) \mid {}^t g J_2 g = J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \det g = 1 \right\}.$$ 

Let $\theta(g) = {}^t \bar{g}^{-1} (g \in G)$ be a Cartan involution of $G$ and $K$ be the set of fixed points of $\theta$. Then $K$ becomes a maximal compact subgroup of $G$ which is isomorphic to the unitary group $U(2)$.

Let $\mathfrak{g} = \{ X \in M_4(\mathbb{R}) \mid J_2 X + {}^t X J_2 = 0 \}$ be the Lie algebra of $G$. If we denote the differential of $\theta$ again by $\theta$, then we have $\theta(X) = -{}^t \bar{X} (X \in \mathfrak{g})$. Let $\mathfrak{t}$ and $\mathfrak{p}$ be the $+1$ and $-1$ eigenspaces of $\theta$ in $\mathfrak{g}$, respectively, and hence

$$\mathfrak{t} = \left\{ X \in \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A, B \in M_2(\mathbb{R}), {}^t A = -A, {}^t B = B \right\},$$

$$\mathfrak{p} = \left\{ X \in \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A, B \in M_2(\mathbb{R}), {}^t A = A, {}^t B = B \right\}.$$
Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Of course, $\mathfrak{k}$ is the Lie algebra of $K$ which is isomorphic to the unitary algebra $\mathfrak{u}(2)$.

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbb{C}} = \mathfrak{l} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{l}$. Let $\mathfrak{h}$ be a compact Cartan subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{h} = \left\{ H(\theta_1, \theta_2) = \begin{pmatrix} \theta_1 & & \\ -\theta_1 & \theta_2 \\ & -\theta_2 \end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}.$$ 

Now we identify a linear form $\beta : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ with $(\beta_1, \beta_2) \in \mathbb{C}^{2}$ via $\beta = \beta_1 e_1 + \beta_2 e_2$, where $e_i(H(\theta_1, \theta_2)) = \sqrt{-1} \theta_i$. Then the set of roots $\Delta = \Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ is given by

$$\Delta = \{ \pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1) \}.$$ 

Fix a positive root system $\Delta^{+} = \{ (2,0), (0,2), (1,1), (1,-1) \}$, and put $\Delta^{+}_{c}$ and $\Delta^{+}_{n}$ the set of compact and non-compact positive roots, respectively. Then

$$\Delta^{+}_{c} = \{ (1,-1) \}, \quad \Delta^{+}_{n} = \{ (2,0), (0,2), (1,1) \}.$$ 

If we denote the root space for $\beta \in \Delta$ by $\mathfrak{g}_{\beta}$, then we have a decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{+} \oplus \mathfrak{p}_{-}$ with $\mathfrak{p}_{+} = \sum_{\beta \in \Delta^{+}_{c}} \mathfrak{g}_{\beta}$ and $\mathfrak{p}_{-} = \sum_{\beta \in \Delta^{+}_{n}} \mathfrak{g}_{\beta}$.

Put $P_J$ the Jacobi maximal parabolic subgroup of $G$ with the Langlands decomposition $P_J = M_J A_J N_J$, where

$$M_J = \left\{ \begin{pmatrix} \varepsilon & b \\ a & \varepsilon \\ c & d \end{pmatrix} \right\} \varepsilon \in \{ \pm 1 \}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \right\} \simeq \{ \pm I \} \times SL(2, \mathbb{R}),$$

$$N_J = \left\{ n(x, y; z) = \begin{pmatrix} 1 & y & x \\ 1 & 1 & 1 \\ 1 & -y & 1 \end{pmatrix} \right\}, \quad x, y, z \in \mathbb{R}.$$ 

and $A_J = \{ \text{diag}(a, 1, a^{-1}, 1) \mid a > 0 \}$. Remark that the unipotent radical $N_J$ of $P_J$ is isomorphic to the 3-dimensional Heisenberg group $\mathcal{H}_1$. The Levi part $M_J A_J$ of $P_J$ acts on $N_J$ via the conjugate action, and $M_J$ gives the centralizer of the center $Z(N_J) = \{ n(0, 0; z) \mid z \in \mathbb{R} \} \simeq \mathbb{R}$ of $N_J$ in $M_J A_J$. Now we define the Jacobi group $R_J$ by the semidirect product $M_J^{\circ} \ltimes N_J \simeq SL(2, \mathbb{R}) \ltimes \mathcal{H}_1$, where $M_J^{\circ} \simeq SL(2, \mathbb{R})$ is the identity component of $M_J$.

2.2. Representations. First we investigate the irreducible unitary representations of the Jacobi group $R_J$. Since $Z(R_J) = Z(N_J) \simeq \mathbb{R}$, the central characters of elements in $\hat{R}_J$ and $\hat{N}_J$ are parametrized by the real numbers. Then we call an irreducible unitary representation of $R_J$ and $N_J$ of type $m$ if its central character is of the form $z \mapsto e^{2\pi \sqrt{-1} m z}$ with $m \in \mathbb{R}$. Let $\nu \in \hat{N}_J$ of type $m$. According to the
theorem of Stone-von Neumann (cf. Corwin-Greenleaf [1; pp.46-47, 51-52]), \( \nu \) is a character if \( m = 0 \) and \( \nu \) is infinite dimensional if \( m \neq 0 \). More over \( \nu \) of type \( m \neq 0 \) is uniquely determined by \( m \) up to unitary equivalence. Now we fix an irreducible unitary representation \((\nu_m, U_m)\) of \( N_J \) of type \( m \neq 0 \). From the theory of the Weil representation, \((\nu_m, U_m)\) can be extended to a continuous true projective unitary representation \((\tilde{\nu}_m, U_m)\) of \( R_J \) by \( \tilde{\nu}_m(\tilde{n}) = W_m(g)\nu_m(n) \) for \( \tilde{n} = g \cdot n \in M_J \ltimes N_J \). With the Weil representation \( W_m \) on \( M_J \). Here \( \tilde{\nu}_m \) has a factor set \( \alpha \) which is a proper 2-cocycle.

**Lemma 2.1.** (Satake [7; Appendix I, Proposition 2]) Let \( \tilde{\nu}_m \) \((m \neq 0)\) as above. For every irreducible projective unitary representation \( \pi \) of \( M_J \) with factor set \( \alpha^{-1} \), put \( \rho(\tilde{n}) = \pi(g) \otimes \tilde{\nu}_m(\tilde{n}) \) for \( \tilde{n} = g \cdot n \in M_J \ltimes N_J \). Then \( \rho \) is an irreducible unitary representation of \( R_J \). Conversely, all irreducible unitary representations of \( R_J \) of type \( m \neq 0 \) are obtained in this manner. Moreover \( \rho \) is square-integrable iff \( \pi \) is so.

Let \((\rho, F_\rho)\) be an irreducible unitary representation of \( R_J \) of type \( m \neq 0 \). From the above lemma, we can regard \((\rho, F_\rho) \in \hat{R}_J\) as a tensor product representation \((\pi_1 \otimes \tilde{\nu}_m, \mathcal{W}_{\pi_1} \otimes U_m)\). Here, if we write \( \tilde{M}_J \) for the double cover of \( M_J \simeq SL(2, \mathbb{R}) \), \((\tilde{\nu}_m, U_m)\) is a unitary representation of \( \tilde{M}_J \ltimes N_J \) which is extended from \((\nu_m, U_m) \in N_J \) as above and \((\pi_1, \mathcal{W}_{\pi_1})\) is a unitary representation of \( \tilde{M}_J \) which does not factor through \( M_J \). On the other hand, the unitary dual of \( \tilde{M}_J \) is given as follows.

**Proposition 2.2.** (cf. Gelbert[2; Lemma 4.1, 4.2]) The following representations exhaust a set of representatives for the equivalence classes of irreducible unitary representations of \( SL(2, \mathbb{R}) \).

1. (unitary principal series) \( \mathcal{P}_s^\tau \), \( s \in \sqrt{-1} \mathbb{R} \), \( \tau = 0, 1, 1/2 \) except for the case \((s, \tau) = (0, 1)\).
2. (complementary series) \( \mathcal{C}_s^\tau \), \( 0 < s < 1 \) for \( \tau = 0, 1 \) and \( 0 < s < 1/2 \) for \( \tau = \pm 1/2 \).
3. ((limit of) discrete series) \( \mathcal{D}_k^\pm \), \( k \in 1/2 \mathbb{Z}_{\geq 2} \).
4. (quotient representation) \( \mathcal{D}_{1/2}^\pm, \mathcal{D}_{1/2}^+ \).
5. The trivial representation of \( SL(2, \mathbb{R}) \).

In the above, the representations \( \mathcal{P}_s^\tau, \mathcal{C}_s^\tau \) for \( \tau = 0, 1 \), \( \mathcal{D}_k^\pm \) for \( k \in \mathbb{Z}_{\geq 1} \) and (5) factor through \( SL(2, \mathbb{R}) \), and the otherwise not.

Hence we take as \((\pi_1, \mathcal{W}_{\pi_1})\) one of the irreducible unitary representations \( \mathcal{P}_s^\tau, \mathcal{C}_s^\tau \) with \( \tau = \pm 1/2 \) and \( \mathcal{D}_k^\pm \) with \( k \in 1/2 \mathbb{Z} \setminus \mathbb{Z}, k \geq 1/2 \).

**Remark 2.3.** The Weil representation \( W_m \) considered as the representation of \( \tilde{M}_J \) has the following irreducible decomposition:

\[
W_m = \begin{cases} 
\mathcal{D}_{1/2}^+ + \mathcal{D}_{1/2}^-, & \text{if } m > 0, \\
\mathcal{D}_{1/2}^- + \mathcal{D}_{1/2}^+, & \text{if } m < 0.
\end{cases}
\]

Next, we treat the irreducible unitary representations of \( K \). Since \( \Delta_+ \) is also a positive system of \( \Delta(\mathfrak{k}_C, \mathfrak{h}_C) \), then the set of the \( \Delta_+ \)-dominant weights, and thus
\( \hat{K} \), is parametrized by the set

\[
\Lambda = \{ (\lambda = (\lambda_1, \lambda_2) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \}
\]

(cf. Knapp[4; Theorem 4.28]). We denote by \((\tau(a), V_{a})\) the element of \( \hat{K} \) corresponding to \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \). Here \( \dim V_{\lambda} = d_{\lambda} + 1 \) with \( d_{\lambda} = \lambda_1 - \lambda_2 \).

Both of \( p_{\pm} \) become \( K \)-modules via the adjoint representation of \( K \), and we have isomorphisms \( p_{+} \simeq V_{(2, 0)} \) and \( p_{-} \simeq V_{(0, -2)} \). For a given irreducible \( K \)-module \( V_{\lambda} \) with the parameter \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \), the tensor product \( K \)-modules \( V_{\lambda} \otimes p_{+} \) and \( V_{\lambda} \otimes p_{-} \) have the irreducible decompositions

\[
V_{\lambda} \otimes p_{+} \simeq \bigoplus_{\beta \in \Delta_{n}^{+}} V_{\lambda+\beta}, \quad V_{\lambda} \otimes p_{-} \simeq \bigoplus_{\beta \in \Delta_{n}^{-}} V_{\lambda-\beta}.
\]

For each \( \beta \in \Delta_{n}^{+} \), let \( P_{\beta} : V_{\lambda} \otimes p_{+} \to V_{\lambda+\beta} \) and \( P_{-\beta} : V_{\lambda} \otimes p_{-} \to V_{\lambda-\beta} \) be the projectors into the irreducible factors of \( V_{\lambda} \otimes p_{\pm} \).

In this note, we consider the following two series of representations of \( G \): one is the principal series induced from \( P_{J} \), and the other is the discrete series. We explain these representations in the remaining of this section.

Let \( \sigma = \epsilon, D \) be a representation of \( M_{J} \simeq \{ \pm I \} \times SL(2, \mathbb{R}) \) with a character \( \epsilon : \{ \pm I \} \to \mathbb{C}^{\times} \) and a discrete series representation \( D = D_{n}^\pm (n \in \mathbb{Z}_{\geq 2}) \) of \( SL(2, \mathbb{R}) \), and take a quasi-character \( \nu_{z} (z \in \mathbb{C}) \) of \( A_{J} \) such that \( \nu_{z}(\text{diag}(a, 1, a^{-1}, 1)) = a^{z} \). Then we can construct a induced representation \( \text{Ind}_{P_{J}}^{G}(\sigma \otimes \nu_{z} \otimes 1_{N_{J}}) \) of \( G \) from the Jacobi maximal parabolic subgroup \( P_{J} = M_{J}A_{J}N_{J} \) by the usual manner (cf. Knapp[4; Chapter VII]), and call \( \text{Ind}_{P_{J}}^{G}(\sigma \otimes \nu_{z} \otimes 1_{N_{J}}) \) the \( P_{J} \)-principal series representation of \( G \). The following lemma is derived from the Frobenius reciprocity for induced representations.

**Lemma 2.4.** \( \tau_{\lambda} \in \hat{K} \) with the parameter \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \) such that \( \lambda_1 < n \) (resp. \( \lambda_2 > -n \)) does not occur in the \( K \)-type of \( \text{Ind}_{P_{J}}^{G}(\sigma \otimes \nu_{z} \otimes 1_{N_{J}}) \) for \( D = D_{n}^{+} \) (resp. \( D_{n}^{-} \)). The 'corner' \( K \)-types \( \tau_{\lambda} \in \hat{K} \) of \( \text{Ind}_{P_{J}}^{G}(\sigma \otimes \nu_{z} \otimes 1_{N_{J}}) \) with the parameter \( \lambda \in \Lambda \) given below occur with multiplicity one:

1. \( \lambda = (n, n) \) for \( \epsilon(\gamma) = (-1)^{n} \) and \( D = D_{n}^{+} \),
2. \( \lambda = (n, n-1) \) for \( \epsilon(\gamma) = (-1)^{n} \) and \( D = D_{n}^{-} \),
3. \( \lambda = (-n, -n) \) for \( \epsilon(\gamma) = (-1)^{n} \) and \( D = D_{n}^{-} \),
4. \( \lambda = (-n+1, -n) \) for \( \epsilon(\gamma) = (-1)^{n} \) and \( D = D_{n}^{-} \).

Here \( \gamma = \text{diag}(-1, 1, -1, 1) \).

In order to parametrize the discrete series representations of \( G \), we enumerate all the positive root systems compatible to \( \Delta_{c}^{+} \):

1. \( \Delta_{I}^{+} = \{(1, -1), (2, 0), (1, 1), (0, 2)\} \),
2. \( \Delta_{II}^{+} = \{(1, -1), (2, 0), (1, 1), (0, -2)\} \),
3. \( \Delta_{III}^{+} = \{(1, -1), (2, 0), (0, -2), (-1, -1)\} \),
4. \( \Delta_{IV}^{+} = \{(1, -1), (0, -2), (-1, -1), (-2, 0)\} \).
Let \( J \) be a variable running over the set of indices I, II, III, IV, and let us denote the set of non-compact positive roots for the index \( J \) by \( \Delta^+_J = \Delta^+_J - \Delta^+_c \). Define a subset \( \Xi_J \) of \( \Delta^+_c \)-dominant weights by

\[
\Xi_J = \{ \Lambda = (\Lambda_1, \Lambda_2), \Delta^+_c \text{- dominant weight} \mid \langle \Lambda, \beta \rangle > 0, \forall \beta \in \Delta^+_J \}.
\]

The set \( \bigcup_{J=1}^{IV} \Xi_J \) gives the Harish-Chandra parametrization of the discrete series representation of \( G \). Let us write by \( \pi_J \) the discrete series representation of \( G \) with the Harish-Chandra parameter \( \Lambda \in \bigcup_{J=1}^{IV} \Xi_J \). Then \( \pi_\Lambda \) is called the holomorphic discrete series representation if \( \Lambda \in \Xi_I \) and the anti-holomorphic one if \( \Lambda \in \Xi_{IV} \). Moreover if \( \Lambda \in \Xi_{III} \cup \Xi_{III} \), a discrete series representation \( \pi_\Lambda \) is called large (in the sense of Vogan[8]).

The Blattner formula gives the description of the \( K \)-types of \( \pi_\Lambda \). In particular, the minimal \( K \)-type \((\tau_\Lambda, V_\Lambda)\) of \( \pi_\Lambda \) is given by the formula \( \lambda = \Lambda - \rho_c + \rho_n \), where \( \rho_c \) (resp. \( \rho_n \)) is the half sum of compact (resp. non-compact) positive roots in \( \Delta^+_J \).

We call such \( \lambda \) the Blattner parameter of \( \pi_\Lambda \).

3. Fourier-Jacobi type spherical functions

3.1. Radial parts. Let \((\rho, \mathcal{F}_\rho)\) be an irreducible unitary representation of \( R_J \) and let \((\tau, V_\tau)\) be a finite dimensional \( K \)-module. We denote by \( C^\infty_{\rho, \tau}(R_J \backslash G/K) \) the space of smooth functions \( F: G \rightarrow \mathcal{F}_\rho \otimes V_\tau \) with the property

\[
F(rgk) = (\rho(\tau) \otimes \tau(k))^{-1} F(g), \quad (r, g, k) \in R_J \times G \times K.
\]

On the other hand, let \( C^\infty(A_J; \rho, \tau) \) be the space of smooth functions \( \varphi: A_J \rightarrow \mathcal{F}_\rho \otimes V_\tau \) satisfying

\[
(\rho(m) \otimes \tau(m)) \varphi(a) = \varphi(a), \quad m \in R_J \cap K = M_J^\mathsf{o} \cap K, \quad a \in A_J.
\]

Because of an Iwasawa decomposition of \( G \), we have \( G = R_J A_J K \). Also we remark that all elements in \( M_J^\mathsf{o} \cap K \) are commutative with \( a \in A_J \). Then the restriction to \( A_J \) gives a linear map from \( C^\infty_{\rho, \tau}(R_J \backslash G/K) \) to \( C^\infty(A_J; \rho, \tau) \), which is injective. For each \( f \in C^\infty_{\rho, \tau}(R_J \backslash G/K) \), we call \( f|_{A_J} \in C^\infty(A_J; \rho, \tau) \) the radial part of \( f \), where \( f|_{A_J} \) means the restriction to \( A_J \).

Let \((\tau', V_{\tau'})\) be also a finite dimensional \( K \)-module. For each \( \mathsf{C} \)-linear map \( u: C^\infty_{\rho, \tau}(R_J \backslash G/K) \rightarrow C^\infty_{\rho, \tau}(R_J \backslash G/K) \), we have a unique \( \mathsf{C} \)-linear map \( \mathcal{R}(u): C^\infty(A_J; \rho, \tau) \rightarrow C^\infty(A_J; \rho, \tau') \) with the property \( (uf)|_{A_J} = \mathcal{R}(u)(f|_{A_J}) \) for \( f \in C^\infty_{\rho, \tau}(R_J \backslash G/K) \). We call \( \mathcal{R}(u) \) the radial part of \( u \).

3.2. Fourier-Jacobi type spherical functions. Let \((\rho, \mathcal{F}_\rho)\) be as above and consider a \( C^\infty \)-induced representation \( C^\infty \text{Ind}^{G}_{R_J}(\rho) \) with the representation space

\[
C^\infty_{\rho}(R_J \backslash G) = \{ F: G \rightarrow \mathcal{F}_\rho, \ C^\infty \mid F(rg) = \rho(r)F(g), \ (r, g) \in R_J \times G \}
\]
on which \( G \) acts by the right translation. Then \( C^\infty_{\rho}(R_J \backslash G) \) becomes a smooth \( G \)-module and a \((\mathfrak{g}_C, K)\)-module naturally. Moreover let \((\tau, V_\tau) \in \hat{K} \) and take an
irreducible Harish-Chandra module $\pi$ of $G$ with the $K$-type $\tau^*$, where $\tau^*$ is the contragredient representation of $\tau$. Now we consider the intertwining space

$$\mathcal{I}_{\rho,\pi} := \text{Hom}_{(gC, K)}(\pi, C^\infty \text{Ind}_{J}^{G} (\rho))$$

between $(gC, K)$-modules and its restriction to the $K$-type $\tau^*$ of $\pi$.

Let $i : \tau^* \rightarrow \pi|_{K}$ be a $K$-equivariant map and let $i^*$ be the pullback via $i$. Then the map

$$\mathcal{I}_{\rho,\pi} \xrightarrow{i^*} \text{Hom}_{K}(\tau^*, C^\infty_{\rho}(R_{J}\backslash G)) \simeq C^\infty_{\rho, \tau^*}(R_{J}\backslash G/K)$$

gives the restriction of $T \in \mathcal{I}_{\rho,\pi}$ to the $K$-type $\tau^*$ and we denote the image of $T$ in $C^\infty_{\rho, \tau^*}(R_{J}\backslash G/K)$ by $T_i$. Now the space $\mathcal{J}_{\rho,\pi}(\tau)$ of the algebraic Fourier-Jacobi type spherical functions of type $(\rho, \pi; \tau)$ on $G$ is defined by

$$\mathcal{J}_{\rho,\pi}(\tau) := \bigcup_{i \in \text{Hom}_{K}(\tau^*, \pi|_{K})} \{ T_i \mid T \in \mathcal{I}_{\rho,\pi} \}.$$  

Moreover put

$$\mathcal{J}_{\rho,\pi}^\circ(\tau) = \{ f \in \mathcal{J}_{\rho,\pi}(\tau) \mid f|_{A_{J}}(\text{diag}(a, 1, a^{-1}, 1)) \text{ is of moderate growth as } a \to \infty \}.$$  

We call $f \in \mathcal{J}_{\rho,\pi}^\circ(\tau)$ a Fourier-Jacobi type spherical functions of type $(\rho, \pi; \tau)$.

In this note, we investigate the space $\mathcal{J}_{\rho,\pi}^\circ(\tau)$ for the following triplet $(\rho, \pi; \tau)$: As $\pi \in \hat{G}$ and $\tau^* \in \hat{K}$, we take either the $P_{J}$-principal series representation and the corner $K$-type or the discrete series representation and the minimal $K$-type, and also as $\rho \in \hat{R}_{J}$ the one with the non-trivial central character, i.e. of type $m \neq 0$.

4. Differential equations

4.1. Differential operators. In this subsection, we introduce some differential operators acting on $C^\infty_{\rho, \tau}(R_{J}\backslash G/K)$.

Take an orthonormal basis $\{X_{i}\}$ of $\mathfrak{p}$ with respect to the Killing form of $g$. Now we define a first order gradient type differential operator

$$\nabla_{\rho, \tau} : C^\infty_{\rho, \tau}(R_{J}\backslash G/K) \to C^\infty_{\rho, \tau \otimes \text{Ad}_{gC}}(R_{J}\backslash G/K)$$

by

$$\nabla_{\rho, \tau} f = \sum_{i} R_{X_{i}} f \otimes X_{i}, \quad f \in C^\infty_{\rho, \tau}(R_{J}\backslash G/K),$$

where

$$R_{X} f(g) = \frac{d}{dt} f(g \cdot \exp(tX)) \bigg|_{t=0}, \quad X \in gC, \ g \in G.$$  

This differential operator $\nabla_{\rho, \tau}$ is called the Schmid operator. Then $\nabla_{\rho, \tau}$ can be decomposed as $\nabla_{\rho, \tau}^+ \oplus \nabla_{\rho, \tau}^-$ with $\nabla_{\rho, \tau}^\pm : C^\infty_{\rho, \tau}(R_{J}\backslash G/K) \to C^\infty_{\rho, \tau \otimes \text{Ad}_{p_{C, \pm}}(R_{J}\backslash G/K)}$ corresponding to the decomposition $p_{C} = p_{+} \oplus p_{-}$. For each $\beta \in \Delta_{n}^+$, the shift operator $\nabla_{\rho, \tau_{\lambda}}^{\pm \beta} : C^\infty_{\rho, \tau_{\lambda}}(R_{J}\backslash G/K) \to C^\infty_{\rho, \tau_{\lambda} \pm \beta}(R_{J}\backslash G/K)$ is defined as the composition of
\[ \nabla_{\rho,\tau}^{\pm\beta} \text{ with the projector } P^{\pm\beta} \text{ from } V_{\tau} \otimes p_{\pm} \text{ into the irreducible component } V_{\tau \lambda \pm \beta}; \]
\[ \nabla_{\rho,\tau}^{\pm\beta} \equiv (1_{F_{\rho}} \otimes P^{\pm\beta}) \nabla_{\rho,\tau}^{\pm}. \]

On the other hand, the Casimir element \( \Omega \) is defined by \( \Omega = \sum X_i - \sum Y_j \), where \( \{Y_j\} \) is an orthonormal basis of \( \mathfrak{f} \) with respect to the Killing form of \( \mathfrak{g} \). It is well known that \( \Omega \) is in the center \( Z(\mathfrak{g}_C) \) of the universal enveloping algebra of \( \mathfrak{g}_C \).

4.2. Differential equations. In this subsection, we consider the system of differential equations satisfied by the Fourier-Jacobi type spherical functions.

First we discuss the case of the \( P_J \)-principal series representation \( \pi \in \hat{G} \) and the corner \( K \)-type \( \tau^* \). It is well known that the Casimir element \( \Omega \in Z(\mathfrak{g}_C) \) acts on \( \pi \), hence on \( J_{\rho,\pi}(\tau) \), as the scalar operator \( \chi_{\Omega} \) (cf. Knapp[4; Corollary 8.14]). Let \( \pi = \text{Ind}_{P_J}^{G}(\sigma \otimes \nu_z \otimes 1_{N_J}) \) with data \( \sigma = (\epsilon, D_n^+) \), \( \epsilon(\gamma) = (-1)^n \), and \( \tau^* = \tau^*_h \) be the corner \( K \)-type of \( \pi \), i.e. \( \lambda = (-n, -n) \). Since \( \tau^* \lambda = \tau^* \lambda_{(2,2)} \in \hat{K} \) does not occur in the \( K \)-types of \( \pi \) from Lemma 2.4, an element in \( J_{\rho,\pi}(\tau) \) is annihilated by the action of the composition of the shift operators

\[ \nabla_{\rho,\tau}^{(0,2)} \circ \nabla_{\rho,\tau}^{(2,0)} : C_{\rho,\tau}^{\infty}(R_J \backslash G/K) \to C_{\rho,\tau}^{\infty}(R_J \backslash G/K). \]

Hence we have a system of differential equations satisfied by \( f \) in \( J_{\rho,\pi}(\tau); \)

\[ \begin{align*}
\Omega f &= \chi_{\Omega} f, \\
\nabla_{\rho,\tau}^{(0,2)} \circ \nabla_{\rho,\tau}^{(2,0)} f &= 0.
\end{align*} \]

Let \( \pi = \text{Ind}_{P_J}^{G}(\sigma \otimes \nu_z \otimes 1_{N_J}) \) with data \( \sigma = (\epsilon, D_n^+) \), \( \epsilon(\gamma) = (-1)^n \), and \( \tau^* = \tau^*_h \) be the corner \( K \)-type of \( \pi \), i.e. \( \lambda = (-n+1, -n) \). Since \( \tau^* \lambda_{(1,1)} = \tau^* \lambda_{(2,n-1)} \in \hat{K} \) does not occur in the \( K \)-types of \( \pi \) from Lemma 2.4, therefore an element in \( J_{\rho,\pi}(\tau) \) vanishes by the action of the shift operator

\[ \nabla_{\rho,\tau}^{(1,1)} : C_{\rho,\tau}^{\infty}(R_J \backslash G/K) \to C_{\rho,\tau}^{\infty}(R_J \backslash G/K). \]

Hence we have a system of differential equations satisfied by \( f \) in \( J_{\rho,\pi}(\tau); \)

\[ \begin{align*}
\Omega f &= \chi_{\Omega} f, \\
\nabla_{\rho,\tau}^{(1,1)} f &= 0.
\end{align*} \]

For the case with the data \( \sigma = (\epsilon, D_n^-) \), we have similar systems of equations from the Casimir operator and the shift operators.

Let \( \pi = \pi^{\Lambda} \) be a discrete series representation of \( G \) with the Harish-Chandra parameter \( \Lambda \in \mathbb{Z}_J \) and \( \tau^* = \tau^*_{\lambda} \in \hat{K} \) be the minimal \( K \)-type of \( \pi \). Now we refer the following proposition which enables us to identify the intertwining space \( I_{\rho,\pi} \) with a solution space of differential equations for any \( \rho \in \hat{K}_J \).

**Proposition 4.1.** (Yamashita [9; Theorem 2.4]) Let \( \pi = \pi^{\Lambda} \in \hat{G} \) and \( \tau^* = \tau^*_{\lambda} \in \hat{K} \) be as above. Then we have a linear isomorphism

\[ I_{\rho,\pi} \simeq \bigcap_{\beta \in \Delta^J_{\rho,\tau}} \ker(\nabla^{-\beta}_{\rho,\tau}) \subset C_{\rho,\tau}^{\infty}(R_J \backslash G/K) \]
for any $\rho \in \hat{R}_J$. In particular, 
\[ J_{\rho,\pi}(\tau) = \{ F \in C^\infty_c(R_J \backslash G/K) \mid \nabla_{\rho,\tau}^{-\beta} F = 0, \ \forall \beta \in \Delta_{J,n}^+ \}. \]

Here the index $J^*$ means IV, III, II and I for $J = I$, II, III and IV, respectively.

5. Result

Solving the systems of the differential equations given by (4.1), (4.2) and Proposition 4.1, we obtain the following theorem.

**Theorem 5.1.** Let $\pi$ be a $P_J$-principal series representation (resp. a discrete series representation) of $G = Sp(2, \mathbb{R})$ and $\tau^*$ be the 'corner' $K$-type (resp. the minimal $K$-type) of $\pi$. For each irreducible unitary representation $\rho$ of $R_J$ of type $m \neq 0$, we have
\[ \dim J_{\rho,\pi}^o(\tau) \leq 1. \]

Moreover the radial parts of the functions in $J_{\rho,\pi}^o(\tau)$ are expressed by the Meijer's $G$-function $G_{2,3}^{3,0}(x \mid a_{1,2} \mid b_{1,2,3})$ or more degenerate similar functions.

Here the Meijer's $G$-function $G_{2,3}^{3,0}(x) = G_{2,3}^{3,0}(x \mid a_{1,2} \mid b_{1,2,3})$ with the complex parameters $a_i, b_j (1 \leq i \leq 2, 1 \leq j \leq 3)$ is the many-valued function defined by the integral
\[ G_{2,3}^{3,0}(x) = \frac{1}{2\pi \sqrt{-1}} \int_L \frac{\prod_{j=1}^3 \Gamma(b_j-t)}{\prod_{i=1}^2 \Gamma(a_i-t)} x^t dt \]

of Mellin-Barnes type, where the contour $L$ is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_j-t)$ ($1 \leq j \leq 3$) once in the negative direction. It is known that, up to constant multiple, $G_{2,3}^{3,0}(x)$ is the unique solution of the linear differential equation of 3-rd order
\[ \{ x^3 \frac{d^3}{dx^3} + \alpha_2(x) x^2 \frac{d^2}{dx^2} + \alpha_1(x) x \frac{d}{dx} + \alpha_0(x) \} y = 0 \]

with
\begin{align*}
\alpha_2(x) & = 3 - b_1 - b_2 - b_3 + x, \\
\alpha_1(x) & = (1-b_1)(1-b_2)(1-b_3) + b_1b_2b_3 + (3-a_1 - a_2)x, \\
\alpha_0(x) & = -b_1b_2b_3 + (1-a_1)(1-a_2)x,
\end{align*}

which decays exponentially as $|x| \to \infty$ in $-\frac{3}{2} \pi < \arg x < \frac{1}{2} \pi$ (See the Meijer's original paper [5] for details).

**Remark 5.2.** Let $\pi$ be a holomorphic discrete series representation of $G$ and $\tau^*$ be the minimal $K$-type of $\pi$. Moreover, put $\rho = \pi_1 \otimes \tilde{\nu}_m \in \hat{R}_J$ as in §2. For each $m \neq 0$, there is at most finitely many $\rho$ such that $\dim J_{\rho,\pi}^o(\tau) = 1$, and then the $\pi_1$-factors of such $\rho$'s are the holomorphic discrete series representations of $\tilde{SL}(2, \mathbb{R})$. Moreover, the radial parts of the functions in $J_{\rho,\pi}^o(\tau)$ are expressed by the function of the form $x^p e^{qx}$ for some constant $p, q$. 
REFERENCES


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