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ON THE RUMIN COMPLEX

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As is well known, Rumin introduced his complex for contact manifolds. Obviously, his method is applicable to the strongly pseudo convex boundary case (see [Ru], and also [A-M1]). Namely, let \((M,^0T'')\) be a strongly pseudo convex CR manifold in a complex manifold \(N\), then by Rumin we have a differential complex on \(M\),

\[
\begin{align*}
\Lambda^{n-1}(^0T')^* & \xrightarrow{D} \theta \wedge \Lambda^{n-2}(^0T')^* \wedge^1(^0T'')^* \xrightarrow{\overline{\partial}_b^{(1)}} \theta \wedge \Lambda^{n-2}(^0T')^* \wedge^2(^0T'')^* \\
\xrightarrow{\overline{\partial}_b^{(p-1)}} \theta \wedge \Lambda^{n-2}(^0T')^* \wedge^p(^0T'')^* \xrightarrow{\overline{\partial}_b^{(p)}} \theta \wedge \Lambda^{n-2}(^0T')^* \wedge^{p+1}(^0T'')^*,
\end{align*}
\]

which recovers the Kohn-Rossi cohomology.

\[
\begin{align*}
\text{for } p = 1, \quad & \text{Ker } \partial^{(1)}_b / \text{Im } D \cong H^{(1)}_b(M, \Lambda^{(n-1)}(T')^*), \\
\text{for } p > 1, \quad & \text{Ker } \overline{\partial}_b^{(p)} / \text{Im } \overline{\partial}_b^{(p-1)} \cong H^{(p)}_b(M, \Lambda^{(n-1)}(T')^*),
\end{align*}
\]

where \(D\) is a second order differential operator which is introduced by Rumin, and \(^0T' = \overline{^0T''}\) and \(\theta\) means a contact form (this is a part of Rumin’s result but a typical one. More precisely, see Sect.1 in this paper).

However this differential complex has an essential weak point. Namely, \(\overline{\partial}_b^{(p)}\) has a natural extension to an ambient space \(N\), but it is not clear if the above second order differential operator \(D\) has a natural extension to an operator on \(N\) or not. By this reason, we propose a new complex, based on Rumin complex, which is applicable to several complex variables (see Sect.2).

Sect.1. CR structures and Rumin complex
Let \((M, ^{0}T'')\) be a CR structure. This means that: \(M\) is a real 2n-1 dimensional \(C^\infty\) manifold and \(^{0}T''\) is a complex subbundle of the complexified tangent bundle satisfying:

1) \(^{0}T'' \cap ^{0}T' = 0\), \(\dim_C(C \otimes TM/(^{0}T'' + ^{0}T')) = 1\)

2) \([\Gamma(M, ^{0}T''), \Gamma(M, ^{0}T'')] \subset \Gamma(M, ^{0}T'')\),

where \(^{0}T' = \overline{^{0}T''}\). In this paper we assume more. Namely, we assume that there is a real global \(C^\infty\) vector bundle \(\xi\) satisfying:

3) \(\xi_p \notin ^{0}T''_p + ^{0}T'_p\) for every point \(p\) of \(M\),

For brevity, we use the following notation.

4) \(T' = C \otimes \xi + ^{0}T'\).

Now we set a real one form \(\theta\) by

\[
\theta(\xi) = 1, \\
\theta|_{^{0}T'' + ^{0}T'} = 0.
\]

Let

\[
\Omega = d\theta.
\]

If this 2-form is positive or negative definite, then our CR-structure \((M, ^{0}T'')\) is called strongly pseudo convex. From now on, we assume that our CR is strongly pseudo convex. By using these notations, we can introduce a \(C^\infty\) vector bundle decomposition:

5) \(\wedge^k(C \otimes TM)^* = \sum_{p+q=k-1, p,q \geq 0} \theta \wedge \wedge^p(^{0}T')^* \wedge \wedge^q(^{0}T'')^* \\
\sum_{r+s=k, r,s \geq 0} \wedge^r(^{0}T')^* \wedge \wedge^s(^{0}T'')^*
\]

We fix this decomposition. And we would like to consider a double complex (for the precise definition, see [A4]). For \(u \in \Gamma(M, \theta \wedge \wedge^{p-1}(^{0}T')^* \wedge \wedge^{q-1}(^{0}T'')^*)\), we set an element of \(\Gamma(M, \wedge^p(^{0}T')^* \wedge \wedge^q(^{0}T'')^*)\) by

\[
u \rightarrow (du)_{\wedge^p(^{0}T')^* \wedge \wedge^q(^{0}T'')^*}
\]

**Proposition 1.1.** This map is a bundle map.

(The proof is a direct computation, and more precisely, see Sect.3.)
Proposition 1.2. If $p + q \geq n$, then this map is surjective and especially, if $p + q = n$, then by comparing dimensions, this is isomorphic.

(For the proof, see lemma 3.3 in [A4])

We use the notation $\kappa^p$ for this isomorphic map from $\theta \wedge^p \wedge q^{-1}(0 T'')$ to $\wedge^p(0 T')^* \wedge q(0 T'')^*$, where $q = n - p$. By using this $\kappa^p$, we set an element $\psi_u$ of $\Gamma(M, \theta \wedge \wedge^p \wedge q^{-1}(0 T'')^*)$ by

$$\psi_u = (\kappa^p)^{-1}((\bar{\partial} b u) \wedge \wedge^p (0 T')^*)$$

where $(\bar{\partial} u) \wedge (0 T')^* \wedge \wedge \wedge (0 T')^*$ means the projection of $\bar{\partial} u$ to $\wedge^p(0 T')^* \wedge \wedge \wedge \wedge (0 T')^*$ according to (5). So by the definition of $\psi_u$, our $\psi_u$ includes the first derivative of $u$.

Sect.2. New complex

Now we introduce a new complex. For a simplicity, we discuss only in the case $p = n - 2$, which is quite related to deformation theory. We set

$$H^0 = \{u : u \in \Gamma(M, \wedge^{n-1}(T')^*), (\bar{\partial} b u) \wedge \wedge^{n-1}(0 T')^* = 0\}$$

$$H^1 = \{u : u \in \Gamma(M, \theta \wedge \wedge^{n-2}(0 T')^* \wedge (0 T'')^*), (\bar{\partial} b^{(1)} u) \wedge \wedge^{n-1}(0 T')^* \wedge \wedge (0 T')^* = 0\}$$

$$H^2 = \{u : u \in \Gamma(M, \theta \wedge \wedge^{n-2}(0 T')^* \wedge \wedge (0 T'')^*), (\bar{\partial} b^{(2)} u) \wedge \wedge^{n-1}(0 T')^* \wedge \wedge (0 T')^* = 0\}$$

Then by definition our $(H^i, \bar{\partial} b)$ is a differential complex

$$\begin{array}{ccc}
H^0 & \xrightarrow{\bar{\partial} b} & H^1 \\
\bar{\partial} b^{(1)} & \xrightarrow{} & H^2
\end{array}$$

Of course $H^1$ (resp. $H^2$) is nothing but our $F^{n-2,1}$ (resp. $F^{n-2,2}$) (see [A-M1], [A4]). And we note that our $H^0$ doesn’t come from $C^\infty$ sections of any $C^\infty$ vector bundle on $M$. This is purely a vector space of some $\wedge^{n-1}(0 T')^*$-valued $C^\infty$ sections. We put an $L^2$ norm on these spaces and discuss the Kodaira Hodge type decomposition theorem on $H^1$. For this, we have to show an a priori estimate. And a difficult problem is to compute the adjoint operator of $\bar{\partial} b$. By the definition of $H^0$, $H^0$ is a subspace of

$$\Gamma(M, \wedge^{n-1}(T')^*) = \Gamma(M, \wedge^{n-1}(0 T')^*) + \Gamma(M, \theta \wedge \wedge^{n-2}(0 T'')^*)$$

And in this canonical decomposition of $\Gamma(M, \wedge^{n-1}(T')^*)$, our $H^0$ can be regarded as a graph of the following map.

For $u \in \Gamma(M, \wedge^{n-1}(0 T')^*)$, 

we set
\[ \psi_u \in \Gamma(M, \theta \wedge \wedge^{n-2}(0T')^*) \]
where \( \psi_u \) is introduced in Sect. 1 in this paper. So
\[ H^0 = \{ v : v = u + \psi_u, u \in \Gamma(M, \wedge^{n-1}(0T')^*) \} \subset \Gamma(M, \wedge^{n-1}(T')^*) \]
For this correspondence, we call the graph map \( i \). On \( \Gamma(M, \wedge^{n-1}(T')^*) \), and \( \Gamma(M, \theta \wedge \wedge^{n-2}(0T')^* \wedge \wedge^p(0T''^*)^*), p = 1, 2, \ldots \), we put \( L_2 \) norm and consider the Kodaira Hodge decomposition theorem on \( (H^p, \overline{\partial}_b) \). The problem is to show an a priori estimate. In proving an a priori estimate, we have to compute, explicitly the adjoint operator " on \( H^p \) spaces (namely we have to write down the term of the adjoint operator "), otherwise, it is impossible to obtain an a priori estimate. We discuss this in the next section.

**Sect. 3. The adjoint operators on \( H^p \) spaces**

We consider the projection of \( \Gamma_2(M, \wedge^{n-1}(T')^*) \) to \( \tilde{H}^0 \), where \( \Gamma_2(M, \wedge^{n-1}(T')^*) \) means the \( L_2 \) - completion of \( \Gamma(M, \wedge^{n-1}(0T')^*) \), and \( \tilde{H}^0 \) means the \( L_2 \) closure of \( H^0 \) in \( \Gamma_2(M, \wedge^{n-1}(T')^*) \). We recall the graph map \( i \).

\[
\Gamma(M, \wedge^{n-1}(0T')^*) \xrightarrow{\text{graph map } i} H^0
\]

We use the notation \( A \) for the composition map of this graph map \( i \) and the inclusion map of \( H^1 \) to \( \Gamma(M, \wedge^{n-1}(T')^*) \). So \( A \) is a map from \( \Gamma(M, \wedge^{n-1}(0T')^*) \) to \( \Gamma(M, \wedge^{n-1}(T')^*) \). By the way, if we put a \( L_2 \) norm on \( \Gamma(M, \wedge^{n-1}(0T''^*) \) by :

\[
\|v\|^2_{\Gamma(M, \wedge^{n-1}(0T'')^*)} = \|v\|^2 + \|\psi_v\|^2 \quad \text{(a graph norm )}
\]

our \( i \) is a norm preserving map (almost tautology). Therefore

\[
(i^*iv, w)_{\Gamma(M, \wedge^{n-1}(0T'')^*)} = (iv, iw)
\]

\[
= (v, w) + (\psi_v, \psi_w)
\]

\[
= (v, w)_{\Gamma(M, \wedge^{n-1}(0T'')^*)}
\]

Namely

\[ i^*i = \text{identity on } \Gamma(M, F) \]

Especially,

\[ i^*(v + \psi_v) = v, \text{ for } v \in \Gamma(M, \wedge^{n-1}(0T''^*) \]

Here \( i^* \) means the adjoint operator of \( i \) with respect to this graph norm ( on \( \Gamma(M, \wedge^{n-1}(0T')^*) \), we use the graph norm defined by \( i \), and on \( \Gamma(M, \wedge^{n-1}(T')^*) \), the standard \( L_2 \) is used ). Then, our main theorem is
Main Theorem. The projection map $= i \cdot A^*$ on $\Gamma(M, T')$.

Proof. For this, it suffices to show that:

1. for $w = w_1 + w_2$, which is orthogonal to $H^0$, we have $i \cdot A^* w = 0$,
2. for $u \in \Gamma(M, \theta \wedge \wedge^{n-2}(0T')^*)$, we have $i \cdot A^*(u + \psi_u) = u + \psi_u$.

where

\begin{align*}
w &\in \Gamma(M, \wedge^{n-1}(T')^*), \\
w_1 &\in \Gamma(M, \wedge^{n-1}(0T')^*), \\
w_2 &\in \Gamma(M, \theta \wedge \wedge^{n-2}(0T''')^*).
\end{align*}

For the proof of (1), by the definition of $w = w_1 + w_2$,

\[(w_1, v) + (w_2, \psi_v) = 0 \text{ for } v \in \Gamma(M, \wedge^{n-1}(0T')^*),\]

But this means

\[(w, Av) = 0 \text{ for } v \in \Gamma(M, \wedge^{n-1}(0T')^*).\]

So

\[(A^*w, v) = 0 \text{ for } v \in \Gamma(M, \wedge^{n-1}(0T')^*).\]

So we have (1).

For the proof of (2),

\[(i \cdot A^*(u + \psi_u) - (u + \psi_u), v + \psi_v) = ((u + \psi_u, Av) - (u + \psi_u, u + \psi_u), \]

\[= (u + \psi_u, Av) - (u + \psi_u, u + \psi_u)\]

for $u, v \in \Gamma(M, \wedge^{n-1}(0T')^*)$. This becomes

\[(A^*(u + \psi_u) - u, v) = (u + \psi_u, v + \psi_v) - (u + \psi_u, v + \psi_v)\]

\[= 0.\]

Sect.4. Kodaira – Hodge type decomposition theorem

In order to establish a Kodaira-Hodge type decomposition theorem on $H^1$, we have to show the following two conditions.

1) $H^1 \cap Dom S$ is dense in $\tilde{H}^1$.
2) The following type a priori estimate.

\[\|S\phi\| + \|\partial_{\bar{b}}^{(1)}\phi\| + \|\phi\| \geq C\|\phi\|_{1/2}\]
for \( \phi \in H^1 \cap \text{Dom } S \), where \( S \) means the adjoint operator of \( \partial_b \) in \((H^p, \partial_b)\) complex, and \( C \) is a positive constant. While, by the result in Sect.2, we have

\[
S = \text{the composition of } \partial_b^* \text{ and } \text{the projection operator of } \Gamma_2(M, \wedge^{n-1}(T')^*) \text{ to } H^0,
\]

where \( \partial_b^* \) means the adjoint operator \( \partial_b \) in the standard complex. Because our complex is a subcomplex, this result follows from functional analysis. Namely, \( S \) is nothing but the adjoint operator of \( D \), which Rumin finds. So, by his estimate (1) is now obvious), we have a Kodaira-Hodge type decomposition theorem over \( H^1 \).

**References**


