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Three dimensional hypersurface purely elliptic singularities

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Abstract

Three dimensional hypersurface purely elliptic singularities are classified into three classes according to the shape of their Newton boundaries.

A lot of examples of their defining equations are obtained from the defining equations of hypersurface simple $K3$ singularities. There are at least 95 types for the defining equations of hypersurface purely elliptic singularities of the type $(0,1)$ or $(0,0)$.

1 Introduction

In the theory of normal two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the most reasonable class of singularities after rational double points. They are characterized as two-dimensional purely elliptic singularities of $(0,1)$-type and of $(0,0)$-type, respectively. What are natural generalizations in three-dimensional case of simple elliptic singularities. The notion of a simple $K3$ singularity was defined in [4] as a three-dimensional isolated Gorenstein purely elliptic singularity of $(0,2)$-type. A simple $K3$ singularity is characterized as a normal three-dimensional isolated singularity such that the exceptional set of any $Q$-factorial terminal modification is a three-dimensional $K3$ surface (see [4]). Here we are interested in three-dimensional hypersurface purely elliptic singularities of $(0,i)$-type for $i=0$ or $i=1$. Let $f\in \mathbb{C}[z_0,z_1,z_2,z_3]$ be a polynomial which

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is non-degenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [5], and whose zero locus $X = \{f = 0\}$ in $\mathbb{C}^4$ has an isolated singularity at the origin $0 \in \mathbb{C}^4$. Then the condition for the singularity $(X, x)$ to be a purely elliptic singularity of $(0, 0)$-type is given by a property of the Newton boundary of $\Gamma(f)$ of $f$.

In this paper, we give the method to obtain the principal parts of defining equations, which define three-dimensional hypersurface purely elliptic singularities of $(0, i)$-type for $i = 0$ to $i = 1$.

### 2 Preliminaries

In this section, we recall some definitions and results from [1], [4] and [6].

First we define the plurigenera $\delta_m, m \in \mathbb{N}$, for normal isolated singularities and define purely elliptic singularities. Let $(X, x)$ be a normal isolated singularity in an $n$-dimensional analytic space $X$, and $\pi : (M, E) \to (X, x)$ a good resolution. In the following, we assume that $X$ is a sufficiently small Stein neighbourhood of $x$.

**Definition.** ([6]) Let $(X, x)$ be a normal isolated singularity. For any positive integer $m$,

$$\delta_m(X, x) := \dim_{\mathbb{C}}\Gamma(X - \{x\}, \mathcal{O}(mK))/L^{2/m}(X - \{x\}),$$

where $K$ is the canonical line bundle on $X - \{x\}$.

Then $\delta_m$ is finite and does not depend on the choice of a Stein neighborhood.

**Definition.** ([6]) A singularity $(X, x)$ is said to be purely elliptic if $\delta_m = 1$ for every $m \in \mathbb{N}$.

When $X$ is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there is a nowhere vanishing holomorphic 2-form on $X - \{x\}$ (see [2]). But in higher dimension, purely elliptic singularities are not always quasi-Gorenstein (see [3]).

In the following, we assume that $(X, x)$ is quasi-Gorenstein. Let $E = \bigcup E_i$ be the decomposition of the exceptional set $E$ into its irreducible components, and write

$$K_M = \pi^*K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_J$$

with $m_i \geq 0, m_j \geq 0$. Ishii [1] defined the essential part of the exceptional set $E$ as $E_j = \sum_{j \in J} m_j E_J$, and showed that if $(X, x)$ is purely elliptic, then $m_j = 1$ for all $j \in J$. 
DEFINITION. ([1],[6]) A quasi-Gorenstein purely elliptic singularity \((X, x)\) is of \((0, i)\)-type if \(H^{n-1}(E_J, \mathcal{O}_E)\) consists of the \((0, i)\)-Hodge component \(H^0,i(E_J)\), where

\[
C \cong H^{n-1}(E_J, \mathcal{O}_E) = Gr^0_{F} H^{n-1}(E_J) = \bigoplus_{i=1}^{n-1} H^0,i(E_J)
\]

\(n\)-dimensional quasi-Gorenstein purely elliptic singularities are classified into \(2n\) classes, including the condition that the singularity is Cohen-Macaulay or not.

Next we consider the case where \((X, x)\) is a hypersurface singularity defined by a non-degenerate polynomial \(f = \sum a_\nu z^\nu \in C[z_0, z_1, \ldots, z_n]\) and \(x = 0 \in C^{n+1}\). Recall that the Newton boundary \(\Gamma(f)\) of \(f\) is the union of the compact faces of \(\Gamma_+(f)\), where \(\Gamma_+(f)\) is the convex hull of \(\bigcup_{a_\nu \neq 0} (\nu + \mathbb{R}^{n+1})\) in \(\mathbb{R}^{n+1}\).

For any face \(\Delta\) of \(\Gamma_+(f)\), set \(f_\Delta := \sum \nu \in \Gamma \nu a_\nu z^\nu\). We say \(f\) to be nondegenerate, if

\[
\frac{\partial f_\Delta}{\partial z_0} = \frac{\partial f_\Delta}{\partial z_1} = \cdots = \frac{\partial f_\Delta}{\partial z_n} = 0
\]

has no solution in \((C^*)^{n+1}\) for any face \(\Delta\). Where \(f\) is nondegenerate, the condition for \((X, x)\) to be a purely elliptic singularity of \((0, i)\)-type is given as follows:

**Theorem 2.1** Let \(f\) be a non-degenerate polynomial and suppose \(X = \{ f = 0 \}\) has an isolated singularity at \(x = 0 \in C^{n+1}\).

1. \((X, x)\) is purely elliptic if and only if \((1, 1, \ldots, 1) \in \Gamma(f)\).
2. Let \(n = 3\) and let \(\Delta_0\) be the face of \(\Gamma(f)\) consisting the point \((1, 1, 1, 1)\) in the relative interior of \(\Delta_0\). Then we have
   1. \((X, x)\) is a singularity of \((0, 2)\)-type if and only if \(\text{dim}_R \Delta_0 = 3\).
   2. \((X, x)\) is a singularity of \((0, 1)\)-type if and only if \(\text{dim}_R \Delta_0 = 2\).
   3. \((X, x)\) is a singularity of \((0, 0)\)-type if and only if \(\text{dim}_R \Delta_0 = 1\) or \(\text{dim}_R \Delta_0 = 0\).

3 **Principal parts**

In this section, we give examples of the principal parts of hypersurface purely elliptic singularities of \((0, i)\)-type defined by a nondegenerate polynomial for \(i = 0\) to \(i = 1\).

**Example.** Let \((X, x)\) be the hypersurface purely elliptic singularity

\[
xyzw + x^{5+p} + y^{5+q} + z^{5+r} + w^{5+s} = 0
\]
in $\mathbb{C}^4$. Blow up the point $O = (0,0,0,0)$, let $F$ be the exceptional set, and let $Y$ be the strict transform of $X$. In this case the morphism $\pi : Y \to X$ is the canonical resolution of $X$. The exceptional set $E$ consists of four 2-dimensional projective spaces in $F$, forming a tetrahedron.

**EXAMPLE.** Let $(X,x)$ be the hypersurface purely elliptic singularity
\[
x^2 + y^3 + z^7 + \mu w^{43+s} + xyzw = 0.
\]
in $\mathbb{C}^4$. Blow up the point $O = (0,0,0,0)$ with weight $(21,14,6,1)$, let $F$ be the exceptional set, and let $Y$ be the strict transform of $X$. In this case the morphism $\pi : Y \to X$ is the canonical resolution of $X$. The exceptional set $E$ is a rational surface with a singularity $T_{2,3,7}$ in a weighted projective space $F$, i.e., $\mathbb{P}(21,14,6,1)$.

**EXAMPLE.** Let $(X,x)$ be the hypersurface singularity defined by the equation
\[
x^2 + y^3 + z^7 + \lambda x^6 w^6 + \mu w^{42} + w^{43+s} + xyzw = 0
\]
in $\mathbb{C}^4$. Then the singularity $(X,x)$ is a purely elliptic singularity of $(0,1)$-type.

**EXAMPLE.** Let $(X,x)$ be the hypersurface singularity defined by the equation
\[
x^2 + y^3 + z^7 + \lambda x^6 w^6 + \mu w^{42} + w^{43+s} + xyzw = 0
\]
in $\mathbb{C}^4$. Then we obtain:

1. $\mu \neq 0 \Leftrightarrow (0,2)$-type.
2. $\mu = 0, \lambda \neq 0 \Leftrightarrow (0,1)$-type.
3. $\mu = 0, \lambda = 0 \Leftrightarrow (0,0)$-type.

**References**


