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A GENERALIZATION OF COLEMAN'S ISOMORPHISM

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1. General Notation. Fix a compatible system \((1, \varepsilon_1, \ldots, \varepsilon_n, \ldots)\) of roots of unity, with \(\varepsilon_{n+1}^p = \varepsilon_n\) and \(\varepsilon_1 \neq 1\). If \(K\) is a finite extension of \(\mathbb{Q}_p\) and \(n \in \mathbb{N}\), let \(K_n = K(\varepsilon_n)\) and \(K_\infty = \bigcup_{n \in \mathbb{N}} K_n\). Let also \(\mathcal{G}_K\) be the Galois group \(\text{Gal}(\overline{\mathbb{Q}}_p/K)\) and \(\chi : \mathcal{G}_K \to \mathbb{Z}_p^*\) be the cyclotomic character and denote by \(\mathcal{H}_K \subset \mathcal{G}_K\) its kernel. Finally, let \(\Gamma_K = \mathcal{G}_K/\mathcal{H}_K = \text{Gal}(K_\infty/K)\) and \(\Lambda_K = \mathbb{Z}_p[[\Gamma_K]]\) be the completed group algebra of \(\Gamma_K\).

2. Coleman’s isomorphism. If \(K = \mathbb{Q}_p\) and \(u = (u_n)_{n \in \mathbb{N}}\) is an element of the projective limit of the groups \(\mathcal{O}_{K_n}^*\) with respect to the norm maps, Coleman proved [5] that there exists a unique element \(\text{Col}_u(T)\) of \((\mathbb{Z}_p[[T]])^*\) such that \(\text{Col}_u(\varepsilon_n - 1) = u_n\) for all \(n \in \mathbb{N}\). Now, as \(\text{Col}_u(T) \in (\mathbb{Z}_p[[T]])^*\), its logarithmic derivative has coefficients in \(\mathbb{Z}_p\) and there is a unique measure \(\mu_u\) on \(\mathbb{Z}_p\) such that

\[
\int_{\mathbb{Z}_p} (1 + T)^x \mu_u = (1 + T) \frac{d}{dT} \log(\text{Col}_u(T)).
\]

Restricting this measure to \(\mathbb{Z}_p^*\) and pulling it back to \(\Gamma_K\) using the cyclotomic character gives us a map from \(\lim \mathcal{O}_{K_n}^*\) to \(\Lambda_K\) which is almost an isomorphism and is known as Coleman's isomorphism. Moreover, the measure giving the Kubota-Leopoldt zeta function is the image of the cyclotomic units via this map and so Coleman’s isomorphism can be thought of as a machine producing \(p\)-adic \(L\)-functions out of compatible systems of units.

All this can be thought of as being related to the \(p\)-adic representation \(\mathbb{Q}_p(1)\). It seems therefore interesting to try to generalize as much as possible the results to other \(p\)-adic representations. A big breakthrough has been made by Perrin-Riou [10] in the case where the representation is crystalline and \(K\) unramified over \(\mathbb{Q}_p\) using \(p\)-adic interpolation of the exponentials of Bloch-Kato [1] for the twists of the representation by powers of the cyclotomic character. Her construction has been refined by Kato-Kurihara and Tsuji in their work on trivial zeroes of \(p\)-adic \(L\)-functions and generalized to the case of de Rham representations in [6]. As explained below, the theory of \((\varphi, \Gamma)\)-modules introduced by Fontaine [7] gives such a generalization without any restriction on the representation.

3. The Iwasawa module attached to a \(p\)-adic representation. Define

\[
H^1_{\text{Iw}}(K, V) = H^1(K, \Lambda_K \otimes V).
\]

This paper is a short summary of the talk I gave at the conference and I would like to take the opportunity to thank the organizers for their invitation.
One can view $\Lambda_K \otimes V$ as the space of measures on $\Gamma_K$ with values in $V$ which makes it possible to define maps

$$H^1_{\text{Iw}}(K, V) \rightarrow H^1(K_n, V(k))$$

$$\mu \rightarrow \int_{\Gamma_{K_n}} \chi(x)^k \mu$$

for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. If $T$ is a $\mathbb{Z}_p$-lattice in $V$ which is stable under the action of $\mathcal{G}_K$, one can show, using Shapiro’s lemma, that the map

$$H^1_{\text{Iw}}(K, V) \rightarrow \mathbb{Q}_p \otimes \left( \lim_{\varprojlim} H^1(K_n, T(k)) \right)$$

$$\mu \rightarrow \left( \ldots, \int_{\Gamma_{K_n}} \chi(x)^k \mu, \ldots \right)$$

is an isomorphism for all $k \in \mathbb{Z}$ (the inverse limit above is taken with respect to corestriction maps). If $V = \mathbb{Q}_p(1)$, Kummer’s theory gives us a natural map from $K_n^*$ to $H^1(K_n, \mathbb{Z}_p(1))$ and, taking inverse limits, a map

$$\delta : \lim_{\varprojlim} \mathcal{O}_{K_n}^* \rightarrow H^1_{\text{Iw}}(K, \mathbb{Q}_p(1)).$$

4. $(\varphi, \Gamma)$-modules and Coleman’s isomorphism. The theory of $(\varphi, \Gamma)$-modules attaches to a $p$-adic representation $V$ a module $D(V)$ with commuting actions of $\Gamma_K$ and a Frobenius endomorphism $\varphi$. One of the nice features of this theory is that it is possible to reconstruct $V$ from $D(V)$ which is a priori a simpler object. One natural problem is therefore to read directly on $D(V)$ the properties of $V$. One of the things that one can recover in this way is the Galois cohomology of $V$ (cf. [8]). Using these results, it is possible to construct (cf. [3]) a natural map $\text{Exp}^* : H^1_{\text{Iw}}(K, V) \rightarrow D(V)$.

To relate the above construction to Coleman’s, let $\mathcal{B}_{\mathbb{Q}_p}$ be the ring of Laurent series $x = \sum_{n \in \mathbb{Z}} a_n \pi^n$ where $a_n$ is a bounded sequence of elements of $\mathbb{Q}_p$ going to 0 when $n$ goes to $-\infty$. This ring is given an action of $\varphi$ and $\Gamma$ via the formulae

$$\gamma(\pi) = (1 + \pi)^{\chi(\gamma)} - 1 \text{ and } \varphi(\pi) = (1 + \pi)^p - 1.$$

Now, if $K = \mathbb{Q}_p$ and $V = \mathbb{Q}_p(1)$, then $D(V)$ is the $\mathcal{B}_{\mathbb{Q}_p}$-module of rank 1 with action of $\Gamma$ twisted by $\chi$ and the following identity holds if $u \in \lim_{\varprojlim} \mathcal{O}_{K_n}^*$

$$\text{Exp}^*(\delta(u)) = (1 + \pi) \frac{d}{d\pi} \log(\text{Col}_u(\pi)),$$

which shows that this map $\text{Exp}^*$ is a direct generalization of Coleman’s isomorphism.

5. Relation with Bloch-Kato exponential map. Using the theory of overconvergent representations and especially the fact that any $p$-adic representation of $\mathcal{G}_K$ is overconvergent [2], it is possible to relate invariants coming from the theory of $(\varphi, \Gamma)$-modules to invariants involving the ring $\mathcal{B}_{\text{dR}}$ of $p$-adic periods. More precisely, the ring $\mathcal{B}_{\text{dR}}$ and the ring $\mathcal{B}$ occurring in the theory of $(\varphi, \Gamma)$-modules are both built up from the ring of Witt vectors of the perfectization of $\mathcal{O}_{\Lambda_{\mathbb{C}_p}/p}$ and overconvergent elements in $\mathcal{B}$ are, by definition, elements $x$ such that $\varphi^{-n}(x)$ has a meaning in $\mathcal{B}_{\text{dR}}$ for $n$ big enough.
Proposition. If $V$ is a de Rham representation of $V$ and $\mu \in H_{\mathrm{Iw}}^{1}(K, V)$, then $\exp^{*}(V)$ is overconvergent and, if $n$ is big enough, the following identity holds in $(B_{\mathrm{dR}}^{+} \otimes V)^{\otimes k}$

$$p^{-n}\varphi^{-n}(\exp^{*}(\mu)) = \sum_{k \in \mathbb{Z}} \exp^{*}\left(\int_{\Gamma_{K_n}} \chi(x)^{-k}\right)$$

(2)

Remark. (i) As mentioned above, $\int_{\Gamma_{K_n}} \chi(x)^{-k}$ is an element of $H^{1}(K_{n}, V(-k))$ and $$\exp^{*} : H^{1}(K_{n}, V(-k)) \rightarrow D_{\mathrm{dR}}(V(-k)) = t^{k}D_{\mathrm{dR}}(V)$$ is the map constructed by Kato [9] and is dual to the exponential of Bloch and Kato [1] for the representation $V^*(1 + k)$.

(ii) The term $\text{CW}_{k,n}(\mu)$ corresponding to $\exp^{*}\left(\int_{\Gamma_{K_n}} \chi(x)^{-k}\right)$ in the sum above can be defined directly from $\exp^{*}(\mu)$ without any reference to $\exp^{*}$ and the maps $\mu \rightarrow \text{CW}_{k,n}(\mu)$ are generalizations of the Coates-Wiles homomorphisms [4]. Thus, formula (2) shows that they are related to Bloch-Kato's exponential maps. This last fact is usually thought of as an explicit reciprocity law.

REFERENCES


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