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# A generalization of Kohnen's estimates for Fourier coefficients of Siegel cusp forms

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The purpose of this article is to show that the main result of [K] is valid for any level.

**Theorem.** *Let  $F$  be a cusp form of integral or half integral weight  $k(> 2)$  with respect to the subgroup  $\Gamma_2(N)$  of  $\mathrm{Sp}_2(\mathbf{Z})$ , where*

$$\Gamma_2(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \mid C \equiv 0 \pmod{N} \right\}.$$

And let its Fourier expansion be given by

$$F(Z) = \sum_T a(T) \exp(2\pi i \operatorname{tr} T \langle Z \rangle),$$

where  $T$  runs over positive definite symmetric half-integral  $2 \times 2$ -matrices. Then we have

$$a(T) \ll_{\varepsilon, F} (\min T)^{5/18+\varepsilon} (\det T)^{(k-1)/2+\varepsilon} \quad (\forall \varepsilon > 0), \quad (1)$$

where  $\min T$  is the smallest positive integer represented by  $T$ .

The idea to prove Theorem is the same as in [K], that is a combination of appropriate estimates for both Fourier coefficients of Jacobi Poincaré series and Petterson norms of Fourier-Jacobi coefficients of Siegel modular forms.

$\mathcal{H}_i$  denotes the Siegel upper half space of degree  $i$  consisting of complex  $i \times i$ -matrices with positive definite imaginary part. We often write

$$Z = X + iY = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} = \begin{pmatrix} u + iv & x + iy \\ x + iy & u' + iv' \end{pmatrix} \in \mathcal{H}_2.$$

For simplicity, we consider only the integral weight case.

**Proposition 1.** *We let  $\Gamma_1^J(N)$  be the Jacobi group which is the semi direct product of  $\Gamma_1(N)$  and  $\mathbf{Z}^2$ , and let  $J_{k,m}^{\text{cusp}}(N)$  be the space of holomorphic Jacobi cusp forms on  $\mathcal{H}_1 \times \mathbf{C}$  of weight  $k$  and index  $m$  with respect to  $\Gamma_1^J(N)$  (cf. e.g. [E-Z]).*

For  $\phi$  in  $J_{k,m}^{cusp}(N)$ , let  $c(n,r)$  be the  $(n,r)$ -th Fourier coefficient of  $\phi$  ( $n, r \in \mathbf{Z}$ ,  $r^2 < 4mn$ ). Put  $D = r^2 - 4mn$ . Then we have

$$c(n,r) \ll_{\varepsilon,k} (m + |D|^{1/2+\varepsilon})^{1/2} \frac{|D|^{k/2-3/4}}{m^{(k-1)/2}} \|\phi\| \quad (\forall \varepsilon > 0)$$

where the constant implied in  $\ll$  depends only on  $\varepsilon$  and  $k$  (not on  $m$ ).

Proof. Let  $P_{n,r} = P_{k,m,n,r}$  be the  $(n,r)$ -th Jacobi Poincaré series in  $J_{k,m}^{cusp}(N)$  characterized by

$$\langle \psi, P_{n,r} \rangle = \lambda_{k,m,D} b_{n,r}(\psi) \quad (\forall \psi \in J_{k,m}^{cusp}(N))$$

where  $b_{n,r}(\psi)$  denotes the  $(n,r)$ -th Fourier coefficients of  $\psi$  and

$$\lambda_{k,m,D} := \frac{1}{2} \Gamma\left(k - \frac{3}{2}\right) \pi^{-k+3/2} m^{k-2} |D|^{-k+3/2}.$$

Then the Cauchy-Schwarz inequality gives

$$|c(n,r)|^2 \leq \lambda_{k,m,D}^{-2} \|\phi\|^2 \langle P_{n,r}, P_{n,r} \rangle = \lambda_{k,m,D}^{-1} b_{n,r}(P_{n,r}) \|\phi\|^2.$$

We can show that the Fourier coefficient of  $P_{n,r}$  as follows (cf. [G-K-Z], p.519);

$$\begin{aligned} b_{n,r}(P_{n,r}) &= 1 + (-1)^k \delta_m(r) + \frac{i^k \pi \sqrt{2}}{\sqrt{m}} \sum_{N|c \geq 1} c^{-3/2} (\exp(r^2/2mc) H_{m,c}^+(n,r) \\ &\quad + (-1)^k \exp(-r^2/2mc) H_{m,c}^-(n,r)) J_{k-3/2}\left(\frac{\pi|D|}{mc}\right), \end{aligned}$$

where

$$\delta_m(r) = \begin{cases} 1 & \text{if } r \equiv 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases},$$

$J_{k-3/2}$  is the modified Bessel function of order  $k - 3/2$ , and

$$H_{m,c}^{\pm}(n,r) := \sum_{x(c), y(c)^*} e_c((mx^2 + rx + n)\bar{y} + ny \pm rx),$$

where  $x$  resp.  $y$  run through  $\mathbf{Z}/c\mathbf{Z}$  resp.  $(\mathbf{Z}/c\mathbf{Z})^*$ ,  $\bar{y}$  denotes an inverse of  $y \pmod{c}$ ,  $e_c(b) := \exp(2\pi i b/c)$  for  $c \in \mathbf{N}$ ,  $b \in \mathbf{Z}/c\mathbf{Z}$ ,  $\varepsilon(y) = 1$  or  $i$  according as  $y \equiv 1 \pmod{4}$  or  $\equiv 3 \pmod{4}$ , and  $\left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right)$  means the Kronecker symbol.  $H_{m,c}^{\pm}(n,r)$  is a certain character sum, which is Gauss sum for  $x$  and Kloosterman sum for  $y$ , and by factorizing  $c$  to prime powers, for  $D := r^2 - 4mn$  we can prove an estimate

$$H_{m,c}^{\pm}(n,r) \ll_{\varepsilon} c^{1+\varepsilon}(D,c) \quad (\forall \varepsilon > 0).$$

From this and the estimate

$$J_{k-3/2}(x) \ll_k \min\{x^{-1/2}, x^{k-3/2}\} \quad (x > 0)$$

(cf. e.g. [B], p.4 and p.74), we easily find

$$b_{n,r}(P_{n,r}) \ll_{\varepsilon,k} 1 + \frac{|D|^{1/2+2\varepsilon}}{m}$$

for any  $\varepsilon > 0$  and complete the proof.

□

To estimate Petterson norm  $\|\phi\|$ , for an analogue of the Rankin convolution series

$$D_{F,F}(s) := \zeta(2s - 2k + 4) \sum_{n \geq 1} \langle \phi_n, \phi_n \rangle n^{-s}$$

where

$$F(Z) = \sum_{n \geq 1} \phi_n(\tau, z) \exp(2\pi i n \tau'),$$

we want to use the following Landau's Theorem;

**Theorem** (Landau-Shintani). *Suppose that*

$$\xi(s) = \sum_{n \geq 1} c(n) n^{-s}, \quad \xi_i(s) = \sum_{n \geq 1} c_i(n) n^{-s} \quad (1 \leq i \leq I)$$

are Dirichlet serieses with non-negative coefficients which converge for  $\operatorname{Re}(s) > \sigma_0$ , have meromorphic continuation to  $\mathbf{C}$  with finitely many poles and satisfy a functional equation

$$\xi^*(\delta - s) = \sum_{i=1}^I \xi_i^*(s)$$

where

$$\xi_i^*(s) = B A^s \prod_{j=1}^J \Gamma(a_j s + b_j) \xi(s) \quad (A \in \mathbf{C}, B \in \mathbf{C}, a_j > 0, b_j \in \mathbf{R}),$$

$$\xi_i^*(s) = B_i A_i^s \prod_{j=1}^J \Gamma(a_j s + b_j) \xi(s) \quad (A_i \in \mathbf{C}, B_i \in \mathbf{C}, a_j \text{ and } b_j \text{ are same as above}).$$

Suppose

$$\kappa := (2\sigma_0 - \delta) \sum_{j=1}^J a_j - \frac{1}{2} > 0.$$

Then we have

$$\sum_{n \leq x} c(n) = \sum_{s: \text{all poles}} \operatorname{Res} \left( \frac{\xi(s)}{s} x^s \right) + O_\eta(x^\eta)$$

for any  $\eta > \eta_0 := \{\delta + \sigma_0(\kappa - 1)\}/(\kappa + 1)$ .

For the proof, see Theorem 3 and its proof in [S-S].

□

The central extension of  $\Gamma_1^J(N)$  by  $\mathbf{Z}$  is embedded into  $\Gamma_2(N)$  via

$$\left( \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu, \kappa \right) \mapsto \begin{pmatrix} a & 0 & b & \mu' \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix}, (\lambda, \mu) = (\lambda', \mu') \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right),$$

and we denote by  $C_N$  the image in  $\Gamma_2(N)$ . Denote the left upper entry of  $Z \in \mathcal{H}_2$  by  $Z_1$ . For a natural number  $N$ ,  $Z \in \mathcal{H}_2$  and  $s \in \mathbf{C}$  with  $\operatorname{Re}(s) \gg 2$  we define a Klingen-Siegel type Eisenstein series

$$E_{s,N}(Z) := \sum_{M \in C_N \setminus \Gamma_2(N)} \left( \frac{\det \operatorname{Im} M \langle Z \rangle}{\operatorname{Im} M \langle Z \rangle_1} \right)^s.$$

It is easily seen that this series is well defined, absolutely convergent, and invariant under the action of  $\Gamma_2(N)$ . We put

$$E_{s,N}^*(Z) := \pi^{-s} \Gamma(s) \zeta(2s) E_{s,N}(Z).$$

By Main Lemma on p.545 in [K-S], we know  $E_{s,1}(Z)$  has a meromorphic continuation to  $\mathbf{C}$ , has only two poles at  $s = 0, 2$  which are simple, and satisfies a functional equation

$$E_{2-s,1}^*(Z) = E_{s,1}^*(Z).$$

By the method of Rankin-Selberg convolution

$$\pi^{-k+2} \langle F E_{s-k+2,N}^*, F \rangle = D_{F,F}^*(s) \quad (2)$$

can be proved, and analytic properties of  $D_{F,F}^*(s)$  follow from those of  $E_{s,N}^*(s)$ . But the functional equations are complicated.

The idea to prove Theorem for any level  $N$  is to write the functional equations satisfied by Eisenstein series as a form

$$E_{2-s,N}^*(Z) = \text{a linear combination of } E_{s,m}^*(Z)$$

where  $m$  is a natural number with  $m|N$ . This is necessary to apply Rankin's method.

**Lemma 1.**  $E_{s,N}(Z)$  has a meromorphic continuation to  $\mathbf{C}$ . Its poles are  $s = 0$  and  $2$ , which are simple. And it satisfies a functional equation

$$E_{2-s,N}^*(Z) = \text{a finite sum of } \frac{\pm n^s}{P(s)} E_{s,m}^*(Z),$$

where  $m, n$  are natural numbers with  $m|N$  and  $P(s)$  is a finite product of  $1 - \tilde{m}^{2(2-s)}$  with  $\tilde{m}|m$ .

Proof. For  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \in \Gamma_2(N)$ , we notice that

$$\frac{\det \operatorname{Im} M \langle Z \rangle}{\operatorname{Im} M \langle Z \rangle_1} = \frac{|Y|}{Y \left[ Z^* \begin{pmatrix} c_4 \\ -c_3 \end{pmatrix} + \begin{pmatrix} d_4 \\ -d_3 \end{pmatrix} \right]}$$

$(Y \begin{bmatrix} a \\ b \end{bmatrix}) := (\bar{a}, \bar{b}) Y \begin{pmatrix} a \\ b \end{pmatrix}$ ,  $Z^*$  means the adjoint matrix of  $Z$  and the mapping

$$\begin{pmatrix} * & * & * & * \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \mapsto (c_3, c_4, d_3, d_4)$$

induces a bijection between

$$C_N \setminus \Gamma_2(N) \text{ and } \{(c_3, c_4, d_3, d_4) \in \mathbf{Z}^4 | \text{primitive and } c_3 \equiv c_4 \equiv 0 \pmod{N}\}.$$

In the following sums,  $c = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$ ,  $d = \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}$  run over  $\mathbf{Z}^2$  under the condition that  $c_3, c_4, d_3, d_4$  are relatively prime. In general, for a square free integer  $m$  and a natural number  $l = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r} \in \mathbf{N}$  (where  $p_1, p_2, \dots, p_r$  are different prime numbers and  $e_i > 0$ ) it holds

$$\begin{aligned} & \frac{1}{l^s} E_{s,m}(lZ) \\ &= \sum_{\substack{(t_c, t_d)=1 \\ c \equiv 0 \pmod{m}}} \frac{|Y|^s}{(Y[Z^*lc + d])^s} \\ &= \left( \sum_{\substack{(t_c, t_d)=1 \\ (l, t_d) \neq 1 \\ c \equiv 0 \pmod{m}}} + \sum_{\substack{(t_c, t_d)=1 \\ (l, t_d)=1 \\ c \equiv 0 \pmod{m}}} \right) \frac{|Y|^s}{(Y[Z^*lc + d])^s} \\ &= \sum_{\substack{(t_c, t_d)=1 \\ d \equiv 0 \pmod{\exists p_i} \\ c \equiv 0 \pmod{m}}} \frac{|Y|^s}{(Y[Z^*lc + d])^s} + \sum_{\substack{(t_c, t_d)=1 \\ c \equiv 0 \pmod{lm}}} \frac{|Y|^s}{(Y[Z^*c + d])^s} \\ &= \sum_i \frac{1}{p_i^{2s}} \sum_{\substack{(t_c, p_i^t d)=1 \\ c \equiv 0 \pmod{m}}} \frac{|Y|^s}{(Y[Z^*(l/p_i)c + d])^s} \\ &\quad - \sum_{i \neq j} \frac{1}{(p_i p_j)^{2s}} \sum_{\substack{(t_c, p_i p_j^t d)=1 \\ c \equiv 0 \pmod{m}}} \frac{|Y|^s}{(Y[Z^*(l/p_i p_j)c + d])^s} \\ &\quad + \dots \\ &\quad + \sum_{\substack{(t_c, t_d)=1 \\ c \equiv 0 \pmod{lm}}} \frac{|Y|^s}{(Y[Z^*c + d])^s} \\ &= \sum_i \frac{1}{p_i^{2s}} \left( \sum_{\substack{(t_c, t_d)=1 \\ c \equiv 0 \pmod{0}}} - \sum_{\substack{(t_c, t_d)=1 \\ c \equiv 0 \pmod{m} \\ c \equiv 0 \pmod{p_i}}} \right) \frac{|Y|^s}{(Y[Z^*(l/p_i)c + d])^s} \\ &\quad - \dots \\ &= \sum_i \frac{1}{(p_i)^{2s}} \{E_{s,m}((l/p_i)Z) - E_{s, \text{l.c.m.}(m, p_i)}((l/p_i)Z)\} \\ &\quad - \sum_{i \neq j} \frac{1}{(l p_i p_j)^s} \{E_{s,m}((l/p_i p_j)Z) - E_{s, \text{l.c.m.}(m, p_i)}((l/p_i p_j)Z) - E_{s, \text{l.c.m.}(m, p_j)}((l/p_i p_j)Z) \\ &\quad \quad + E_{s, \text{l.c.m.}(m, p_i p_j)}((l/p_i p_j)Z)\} \\ &\quad + \dots \end{aligned}$$

$$\begin{aligned}
 &+(-1)^{r-1} \frac{1}{(lp_1p_2 \dots p_r)^s} \{E_{s,m}((l/p_1 \dots p_r)Z) - \sum_i E_{s, \text{l.c.m.}(m,p_i)}((l/p_1 \dots p_r)Z) \\
 &\quad + \sum_{i \neq j} E_{s, \text{l.c.m.}(m,p_i p_j)}((l/p_1 \dots p_r)Z) - \dots + (-1)^r E_{s, \text{l.c.m.}(m,p_1 \dots p_r)}((l/p_1 \dots p_r)Z)\} \\
 &+ E_{s,lm}(Z). \tag{3}
 \end{aligned}$$

We apply (3) for  $m = 1$  and  $l = N$ ; if  $N$  is not square-free the last term is  $E_{s,N}(Z)$ , otherwise the last two terms are  $(-N^{-2s} + 1)E_{s,N}(Z)$ , and in the both cases the rests are  $\pm \tilde{n}^{-s} E_{s,\tilde{m}}(\tilde{l}Z)$  where  $\tilde{l}, \tilde{m}, \tilde{n}$  are natural numbers with  $\tilde{l}\tilde{m}|N$ ,  $\tilde{m} < N$ . Hence for a non-square-free number  $N$  we have

$$E_{s,N}^*(Z) = \text{a finite sum of } \pm n^s E_{s,m}^*(lZ)$$

where  $l, m, n$  are natural numbers with  $lm|N$ ,  $m < N$ , and for a square-free number  $N$  we have

$$(1 - N^{2s})E_{s,N}^*(Z) = \text{a finite sum of the same type as above.}$$

So, by induction on  $N$  we deduce that  $E_{s,N}(Z)$  has a meromorphic continuation to  $\mathbf{C}$ , has poles only at  $s = 0, 2$  and satisfies a functional equation

$$E_{2-s,N}^*(Z) = \text{a finite sum of } \frac{\pm n^s}{P_1(s)} E_{s,m}^*(lZ)$$

where  $l, m, n$  are natural numbers with  $lm|N$  and  $P_1(s)$  is a finite product of  $1 - \tilde{m}^{2(2-s)}$  with  $\tilde{m}|m$ . Now we notice that (3) makes  $l$  smaller, and apply (3) repeatedly in all terms in this right-hand side until  $l$  becomes 1, then finally we get the functional equation in Lemma 1. □

Then we can use Rankin's method and deduce

**Lemma 2.** *Let the notations be as above, and take a natural number  $m$  with  $m|N$ . For  $L \in \Gamma_2$ , we write the Fourier expansions of  $F(L^{-1}\langle Z \rangle)$  as*

$$F(L^{-1}\langle Z \rangle) = \sum_{n \geq 1} \phi_{n,L}(\tau, z) \exp\left(\frac{2\pi i n \tau'}{N}\right).$$

We define a Dirichlet series  $D_{F,F,m}(s)$  as  $\zeta(2s - 2k + 4)$  times

$$\sum_{n \geq 1} \left\{ \sum_{L \in \Gamma_2(N) \setminus \Gamma_2(m)} \int_{\mathcal{F}} |\phi_{n,L}(\tau, z)|^2 \exp\left(-\frac{4\pi n y^2}{vN}\right) v^{k-3} du dv dx dy \right\} n^{-s}$$

where  $\mathcal{F}$  is a fundamental domain  $\Gamma_1^J(m) \setminus \mathcal{H}_1 \times \mathbf{C}$  (so  $D_{F,F,N}(s) = D_{F,F}(s)$ ), and put

$$D_{F,F,m}^*(s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) D_{F,F,m}(s).$$

Then we have

$$\pi^{-k+2} \langle F E_{s-k+2,m}^*, F \rangle = N^s D_{F,F,m}^*(s). \tag{4}$$

□

From (2), (4) and Lemma 1 we have proved

**Proposition 2.**  $D_{F,F,m}(s)$  is a Dirichlet series which has a meromorphic continuation to  $\mathbf{C}$ , possibly has a unique pole at  $s = k$ , and satisfies a functional equation

$$D_{F,F}^*(2k - 2 - s) = D_{F,F,N}^*(2k - 2 - s) = \text{a finite sum of } \frac{\pm n^s}{P(s)} D_{F,F,m}^*(s)$$

where  $m, n$  are natural numbers with  $m|N$  and  $P(s)$  is a finite product of  $1 - \tilde{m}^{2(k-s)}$  with  $\tilde{m}|m$ .

□

Now we can use Landau's Theorem for  $D_{F,F,m}(s)$ 's, because  $D_{F,F,m}(s)/(1 - p^{2(k-s)})$  has non-negative coefficients and has a unique pole at  $s = k$ , hence it converges for  $s > k$ . Therefore we have

$$\sum_{n \leq x} \|\phi_n\|^2 = \left( \operatorname{Res}_{s=k} \frac{D_{F,F}(s)}{s} \right) x^k + O_\varepsilon(x^{k-4/9+\varepsilon}) \quad (\forall \varepsilon > 0)$$

where  $\phi_n$  is the  $n$ -th Fourier-Jacobi coefficient of  $F(Z)$ . Taking  $x = m$  and  $x = m - 1$  and subtracting, we find

$$\|\phi_m\|^2 \ll_{\varepsilon, F} m^{k-4/9+\varepsilon},$$

hence

$$\|\phi_m\| \ll_{\varepsilon, F} m^{k/2-2/9+\varepsilon} \quad (\forall \varepsilon > 0). \quad (5)$$

By Proposition 2 and (5), we obtain

$$c(n, r) \ll_{\varepsilon, k} (m + |D|^{1/2+\varepsilon})^{1/2} |D| m^{5/18+\varepsilon}.$$

Both sides of (1) are invariant if  $T$  is replaced by  ${}^tUTU$  ( $U \in GL_2(\mathbf{Z})$ ). Hence we may assume that

$$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}, \quad m = \min T,$$

so that  $a(T) = c(n, r)$ . By reduction theory we have  $m = \min T \leq \frac{2}{\sqrt{3}}|D|^{1/2}$  and complete the proof of Theorem.

□

*Remark.*

1. When  $N = 1$ , the Rankin convolution series  $D_{F,F}(s)$  is a linear combination of spinor zeta functions of Hecke eigen forms, as shown in [K-S]. In order to deduce estimates for eigenvalues of Hecke operators, we need find a relation between  $D_{F,F,m}(s)$ 's and spinor zeta functions.



2. When we generalize Kohnen's method to higher genus, we should cut  $Z$  as follows;

$$Z = \left( \begin{array}{ccc|c} * & \dots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & * & * \\ \hline * & \dots & * & \tau' \end{array} \right).$$

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