Note on representations of generalized inverse $\ast$-semigroups\(^1\)

Teruo IMAOKA and Takahide OGAWA

Department of Mathematics, Shimane University
Matsue, Shimane 690, Japan

Abstract

The Munn representation of an inverse semigroup $S$, in which the semigroup is represented by isomorphisms between principal ideals of the semilattice $E(S)$, is not always faithful. By introducing a concept of a presemilattice, Reilly considered of enlarging the carrier set $E(S)$ of the Munn representation in order to obtain a faithful representation of $S$ as an inverse subsemigroup of a structure resembling the Munn semigroup $T_{E(S)}$.

The purpose of this paper is to obtain a generalization of the Reilly's results for generalized inverse $\ast$-semigroups.

1 Introduction

A semigroup $S$ with a unary operation $\ast : S \to S$ is called a regular $\ast$-semigroup if it satisfies

(i) $(x^\ast)^\ast = x$,
(ii) $(xy)^\ast = y^\ast x^\ast$,
(iii) $xx^\ast x = x$.

Let $S$ be a regular $\ast$-semigroup. An idempotent $e$ in $S$ is called a projection if it satisfies $e^\ast = e$. For any subset $A$ of $S$, denote the sets of idempotents and projections of $A$ by $E(A)$ and $P(A)$, respectively.

Let $S$ be a regular $\ast$-semigroup. It is called a locally inverse $\ast$-semigroup if, for any $e \in E(S)$, $eSe$ is an inverse subsemigroup of $S$. If $E(S)$ is a normal band, then $S$ is called a generalized inverse $\ast$-semigroup.

Let $S$ and $T$ be regular $\ast$-semigroups. A homomorphism $\phi : S \to T$ is called a $\ast$-homomorphism if $(a\phi)^\ast = a^\ast \phi$. A congruence $\sigma$ on $S$ is called a $\ast$-congruence if

\(^1\)This is the abstract and the details will be published elsewhere
\((a\sigma)^* = a^*\sigma\). A \(*\)-congruence \(\sigma\) on \(S\) is said to be \textit{idempotent-separating} if \(\sigma \subseteq \mathcal{H}\), where \(\mathcal{H}\) is one of the Green's relations. Denote the maximum idempotent-separating \(*\)-congruence on \(S\) by \(\mu_S\) or simply by \(\mu\). If \(\mu_S\) is the identity relation on \(S\), \(S\) is called \textit{fundamental}. The following results are well-known, and we use them frequently throughout this paper.

Result 1.1 [2]. Let \(S\) be a regular \(*\)-semigroup. Then we have the following:

1. \(E(S) = P(S)^2\);
2. for any \(a \in S\) and \(e \in P(S)\), \(a^*ea \in P(S)\);
3. each \(L\)-class and each \(R\)-class have one and only one projection;
4. \(\mu_S = \{(a, b) \in S \times S : a^*ea = b^*eb \text{ and } aea^* = beb^* \text{ for all } e \in P(S)\}\).

For a mapping \(\alpha : A \rightarrow B\), denote the domain and the range of \(\alpha\) by \(d(\alpha)\) and \(r(\alpha)\), respectively. For a subset \(C\) of \(A\), \(\alpha|_C\) means the restriction of \(\alpha\) to \(C\).

As a generalization of the Preston-Vagner representations, one of the authors gave two types of representations of locally [generalized] inverse \(*\)-semigroups in [3], [4] and [5]. In this paper, we follow [5]. A non-empty set \(X\) with a reflexive and symmetric relation \(\sigma\) is called an \(\iota\)-set, and denoted by \((X; \sigma)\). If \(\sigma\) is transitive, that is, if \(\sigma\) is an equivalence relation on \(X\), \((X; \sigma)\) is called a \textit{transitive} \(\iota\)-set.

Let \((X; \sigma)\) be an \(\iota\)-set. A subset \(A\) of \(X\) is called an \(\iota\)-single subset of \((X; \sigma)\) if it satisfies the following condition:

for any \(x \in X\), there exists at most one element \(y \in A\) such that \((x, y) \in \sigma\).

We consider the empty set to be an \(\iota\)-single subset. We remark that if \((X; \sigma)\) is a transitive \(\iota\)-set, a subset \(A\) of \(X\) is an \(\iota\)-single subset if and only if, for \(x, y \in A\), \((x, y) \in \sigma\) implies \(x = y\). A mapping \(\alpha\) in \(I_X\), the symmetric inverse semigroup on \(X\), is called a \textit{partial one-to-one \(\iota\)-mapping} on \((X; \sigma)\) if \(d(\alpha), r(\alpha)\) are both \(\iota\)-single subsets of \((X; \sigma)\), where \(d(\alpha)\) and \(r(\alpha)\) are the domain and the range of \(\alpha\), respectively. Denote the set of all partial one-to-one \(\iota\)-mappings of \((X; \sigma)\) by \(\mathcal{LI}(X; \sigma)\).

For any \(\iota\)-single subsets \(A\) and \(B\) of \((X; \sigma)\), define \(\theta_{A,B}\) by

\[\theta_{A,B} = \{(a, b) \in A \times B : (a, b) \in \sigma\} = (A \times B) \cap \sigma.\]

Since a subset of an \(\iota\)-single subset is also an \(\iota\)-single subset, \(\theta_{A,B} \in \mathcal{LI}(X; \sigma)\). For any \(\alpha, \beta \in \mathcal{LI}(X; \sigma)\), define \(\theta_{\alpha,\beta}\) by \(\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}\), and let \(\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{LI}(X; \sigma)\}\), an indexed set of one-to-one partial functions. Now, define a multiplication \(\circ\) and a unary operation \(\ast\) on \(\mathcal{LI}(X; \sigma)\) as follows:

\[\alpha \circ \beta = \alpha \theta_{\alpha,\beta} \beta \quad \text{and} \quad \alpha^* = \alpha^{-1},\]
where the multiplication of the right side of the first equality is that of $I_X$. Denote $(LI_{(X;\sigma)}, \circ, *)$ by $LI_{(X;\sigma)}(M)$ or simply by $LI_{(X;\sigma)}$. In this paper, we use $LI_{(X;\sigma)}$ rather than $LI_{(X;\sigma)}(M)$.

**Result 1.2** [5]. For any $\iota$-set $(X; \sigma)$, $LI_{(X;\sigma)}$, defined above, is a locally inverse $*$-semigroup. If $(X; \sigma)$ is a transitive $\iota$-set, then $LI_{(X;\sigma)}$ is a generalized inverse $*$-semigroup. In this case, we denote it by $GI_{(X;\sigma)}$ instead of $LI_{(X;\sigma)}$.

Moreover, if $\sigma$ is the identity relation on $X$, then $LI_{(X;\sigma)}$ is the symmetric inverse semigroup $I_X$ on $X$.

We call $LI_{(X;\sigma)} [GI_{(X;\sigma)}]$ the $\iota$-symmetric locally [generalized] inverse $*$-semigroup on the $\iota$-set [the transitive $\iota$-set] $(X; \sigma)$ with the structure sandwich set $M$.

Let $S$ be a regular $*$-semigroup, and define a relation $\Omega$ on $S$ as follows:

$$(x, y) \in \Omega \iff \text{there exists } e \in E(S) \text{ such that } x\rho_e = y,$$

where $\rho_a(a \in S)$ is the mapping of $Sa^*$ onto $Sa$ defined by $x\rho_a = xa$.

**Result 1.3** [5]. Let $S$ be a locally inverse $*$-semigroup. For each $a \in S$, let

$$\rho_a : x \mapsto xa \quad (x \in d(\rho_a) = Sa^*).$$

Then a mapping

$$\rho : a \mapsto \rho_a$$

is a $*$-monomorphism of $S$ into $LI_{(S;\Omega)}(M)$.

For a partial groupoid $X$, if there exist a semilattice $Y$, a partition $\pi : X \sim \sum\{X_e : e \in Y\}$ of $X$ and mappings $\varphi_{e,f} : X_e \rightarrow X_f$ ($e \geq f$ in $Y$) such that

1. for any $e \in Y$, $\varphi_{e,e} = 1_{X_e}$,
2. if $e \geq f \geq g$, then $\varphi_{e,f}\varphi_{f,g} = \varphi_{e,g}$,
3. for $x \in X_e$, $y \in X_f$, $xy$ is defined in $X$ if and only if $x\varphi_{e,f} = y\varphi_{f,e,f}$, and in this case $xy = x\varphi_{e,f}$,

then $X$ is called a **strong $\pi$-groupoid** with mappings $\{\varphi_{e,f} : e, f \in Y, e \geq f\}$, and it is denoted by $X(\pi; Y; \{\varphi_{e,f}\})$ or simply by $X(\pi)$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong $\pi$-groupoid. A subset $A$ of $X$ is called a **$\pi$-singleton subset** of $X(\pi; Y; \{\varphi_{e,f}\})$, if there exists $e \in Y$ such that
$$|A \cap X_f| = \begin{cases} 1 & \text{if } f \in \langle e \rangle, \\ 0 & \text{otherwise,} \end{cases}$$

$$(A \cap X_f) \varphi_{f,g} = A \cap X_g$$

for any $f, g \in \langle e \rangle$ such that $f \geq g$,

where $\langle e \rangle$ is the principal ideal of $Y$ generated by $e$. In this case, we sometimes denote the $\pi$-singleton subset $A$ by $A(e)$. If $A(e)$ is a $\pi$-singleton subset, then $|A \cap X_f| = 1$ for any $f \in \langle e \rangle$. We denote the only one element of $A \cap X_f$ by $a_f$. We remark that, for any $\pi$-singleton subset $A(e)$, $A(e) = \{a_{e\varphi_{e,f}} : f \in \langle e \rangle\}$.

Two $\pi$-singleton subsets $A(e)$ and $B(f)$ are said to be $\pi$-isomorphic to each other, if there exists an isomorphism $\overline{\alpha} : \langle e \rangle \rightarrow \langle f \rangle$ as semilattices. In this case, the mapping $\alpha : A(e) \rightarrow B(f)$ defined by $a_{g\overline{\alpha}} = b_{g\overline{\alpha}}$ ($g \in \langle e \rangle$) is called a $\pi$-isomorphism of $A(e)$ to $B(f)$. It is obvious that $\alpha$ is a bijection of $A(e)$ onto $B(f)$, and hence $\alpha \in \mathcal{I}_X$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong $\pi$-groupoid. Define an equivalence relation $\mathcal{U}$ on $\mathcal{X}$ by

$$\mathcal{U} = \{ (A(e), B(f)) \in \mathcal{X} \times \mathcal{X} : \langle e \rangle \cong \langle f \rangle \text{ (as semilattices)} \}.$$

For $(A(e), B(f)) \in \mathcal{U}$, let $T_{A(e), B(f)}$ be the set of all $\pi$-isomorphisms of $A(e)$ onto $B(f)$, and let

$$T_{X(\pi)} = \bigcup_{(A(e), B(f)) \in \mathcal{U}} T_{A(e), B(f)}.$$

For any $\alpha, \beta \in T_{X(\pi)}$, define a mapping $\theta_{\alpha, \beta}$ as follows:

$$d(\theta_{\alpha, \beta}) = \{a \in r(\alpha) : \text{there exist } e \in Y \text{ and } b \in d(\beta) \text{ such that } a, b \in X_e\},$$

$$r(\theta_{\alpha, \beta}) = \{b \in d(\beta) : \text{there exist } e \in Y \text{ and } a \in r(\alpha) \text{ such that } a, b \in X_e\},$$

$$a_{\theta_{\alpha, \beta}} = b \quad \text{if } r(\alpha) \cap X_e = \{a\} \text{ and } d(\beta) \cap X_e = \{b\}.$$

Then $\theta_{\alpha, \beta} \in T_{X(\pi)}$. Let $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_{X(\pi)}\}$, and define a multiplication $\circ$ and a unary operation $\ast$ on $T_{X(\pi)}$ by

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta},$$

$$\alpha^\ast = \alpha^{-1}.$$

Then $T_{X(\pi)}(\circ, \ast)$ is a regular $\ast$-semigroup. We denote it by $T_{X(\pi)}(\mathcal{M})$. 
Result 1.4 [4]. A regular \( \ast \)-semigroup \( T_{X(\pi)}(\mathcal{M}) \) is a generalized inverse \( \ast \)-semigroup whose set of projections is partially isomorphic to \( X \).

Let \( S \) be a generalized inverse \( \ast \)-semigroup. Hereafter, denote \( E(S) \) and \( P(S) \) simply by \( E \) and \( P \), respectively. Let \( E \sim \sum \{ E_i : i \in I \} \) be the structure decomposition of \( E \), and let \( P_i = P(E_i) \). Then \( \pi : P \sim \sum \{ P_i : i \in I \} \) is a partition of \( P \). For any \( i, j \in I \) \((i \geq j)\), define a mapping \( \varphi_{i,j} : P_i \rightarrow P_j \) by

\[
e\varphi_{i,j} = efe \quad \text{for some (any) } f \in P_j.
\]

Then \( P(\pi; I; \{ \varphi_{i,j} \}) \) is a strong \( \pi \)-groupoid.

Result 1.5 [4]. Let \( S \) be a generalized inverse \( \ast \)-semigroup. For each \( a \in S \), let

\[
\tau_a : e \mapsto a^*ea \quad (e \in d(\tau_a) = P(Sa^*)).
\]

Then a mapping \( \tau : a \mapsto \tau_a \) is a \( \ast \)-homomorphism of \( S \) into \( T_{P(\pi)}(\mathcal{M}) \) such that \( \tau \circ \tau^{-1} = \mu \).

A regular \( \ast \)-subsemigroup \( T \) of a regular \( \ast \)-semigroup \( S \) is said to be \( \mathcal{P} \)-full if \( P(T) = P(S) \).

Result 1.6 [4]. A generalized inverse \( \ast \)-semigroup \( S \) is fundamental if and only if it is \( \ast \)-isomorphic to a \( \mathcal{P} \)-full generalized inverse \( \ast \)-subsemigroup of \( T_{X(\pi)}(\mathcal{M}) \) on a strong \( \pi \)-groupoid \( X(\pi; I; \{ \varphi_{i,j} \}) \) such that \( P(T_{X(\pi)}(\mathcal{M})) \) is partially isomorphic to \( P(S) \).

In § 2, by introducing the concept of partially ordered \( \varrho \)-set \( (X(\preceq); \{ \varphi_\preceq \}) \), we construct a fundamental generalized inverse \( \ast \)-semigroup \( T_{X(\preceq)}(\mathcal{M}) \). Also, we shall see that \( T_{X(\preceq)}(\mathcal{M}) \) has similar properties with \( T_{X(\varpi)}(\mathcal{M}) \), where \( T_{X(\varpi)}(\mathcal{M}) \) has been given by T. Imaoka, I. Inata and H. Yokoyama [4]. And we shall show that two concepts, strong \( \pi \)-groupoids and partially ordered \( \varrho \)-sets, are equivalent.

In § 3, we shall introduce the notion of \( \omega \)-set \( (X(\preceq); \sigma) \), and construct a generalized inverse \( \ast \)-semigroup \( T_{X(\varpi); \sigma}(\mathcal{M}) \). Furthermore, let \( S \) be a generalized inverse \( \ast \)-semigroup with the set of projections \( P \), we shall make two generalized inverse \( \ast \)-semigroups \( T_{P(\varpi)}(\mathcal{M}) \) and \( T_{(S(\varpi)); \Omega}(\mathcal{M}) \), where the former is obtained in § 2, and the latter is constructed in this section. Then we shall show that these three semigroups make a commutative diagram.
2 Fundamental generalized inverse $\ast$-semigroups

2.1 $T_{X(\triangleleft)}(\mathcal{M})$

Let $X(\triangleleft)$ be a partially ordered set, and for each $x \in X$, consider an order-preserving mapping $\phi_x : X \to X$. If a relation $\rho = \{(x, y) \in X \times X : y \phi_x = x, x \phi_y = y\}$ is an equivalence relation on $X$ such that

- (P1) $x \triangleleft y \implies$ for each $y' \in y \rho$, there exists $x' \in x \rho$ such that $x' \triangleleft y'$,
- (P2) a relation $\leq = \{(x \rho, y \rho) \in X/\rho \times X/\rho :$ there exists $x' \in x \rho$ such that $x' \triangleleft y\}$ is a partial order and $X/\rho(\leq)$ is a semilattice,
- (P3) $x_1 \triangleleft y, x_2 \triangleleft y$ and $x_1 \rho \leq x_2 \rho \implies x_1 \leq x_2$,

then $(X(\triangleleft); \{\phi_x\})$ is called a partially ordered $\rho$-set.

Let $(X(\triangleleft); \{\phi_x\})$ be a partially ordered $\rho$-set. Define an equivalence relation $\mathcal{U}$ on $\mathcal{X}$ by

$$\mathcal{U} = \{(\langle a \rangle, \langle b \rangle) \in \mathcal{X} \times \mathcal{X} : \langle a \rangle \simeq \langle b \rangle (\text{order isomorphic})\},$$

where $\mathcal{X}$ is the set of all principal ideals of $(X(\triangleleft); \{\phi_x\})$. For $((a), (b)) \in \mathcal{U}$, let $T_{(a), (b)}$ be the set of all (order) isomorphisms of $(a)$ onto $(b)$, and let

$$T_{X(\triangleleft)} = \bigcup_{((a), (b)) \in \mathcal{U}} T_{(a), (b)}.$$

For any $\alpha, \beta \in T_{X(\triangleleft)}$, define a mapping $\theta_{\alpha, \beta}$ as follows:

$$\theta_{\alpha, \beta} = \{(x, y) \in r(\alpha) \times d(\beta) : (x, y) \in \rho\},$$

where $\rho$ is defined in $(X(\triangleleft); \{\phi_x\})$.

Then $\theta_{\alpha, \beta} \in T_{X(\triangleleft)}$. Let $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_{X(\triangleleft)}\}$, and define a multiplication $\circ$ and a unary operation $\ast$ on $T_{X(\triangleleft)}$ by

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta,$$

$$\alpha^\ast = \alpha^{-1}.$$

Then it is clear that $T_{X(\triangleleft)}(\circ, \ast)$ is a regular $\ast$-subsemigroup of the $\iota$-symmetric generalized inverse $\ast$-semigroup $GI_{X(\triangleleft)}(\mathcal{M})$. Hence it is a generalized inverse $\ast$-semigroup and denoted by $T_{X(\triangleleft)}(\mathcal{M})$.

Let $S$ be a generalized inverse $\ast$-semigroup and $P = P(S)$. We consider $P$ as a partially ordered set with respect to the natural order. Now, we have the following results.
Theorem 2.1 A regular \(\ast\)-semigroup \(T_{X(\leq)}(\mathcal{M})\) is a generalized inverse \(\ast\)-semigroup whose set of projections is order isomorphic to \(X(\leq)\).

Corollary 2.2 A partially ordered set \(X\) is order isomorphic to the set of projections of a generalized inverse \(\ast\)-semigroup if and only if it is a partially ordered \(\varrho\)-set.

2.2 Representations

Let \(S\) be a generalized inverse \(\ast\)-semigroup. Hereafter, denote \(E(S)\) and \(P(S)\) simply by \(E\) and \(P\), respectively. Let \(E \sim \sum \{E_i : i \in I\}\) be the structure decomposition of \(E\), and let \(P_i = P(E_i)\). For any \(e \in P\), define a mapping \(\phi_e : P \to P\) by

\[
f \phi_e = efe.
\]

Let \(e, f \in P\), define a relation \(\leq\) on \(P\) by

\[
e \leq f \iff e = fef,
\]

that is, \(\leq\) is the restriction of natural order on \(S\) to \(P\).

Lemma 2.3 The set \((P(\leq); \{\phi_e\})\), defined above, is a partially ordered \(\varrho\)-set.

Now, we can consider the generalized inverse \(\ast\)-semigroup \(T_{P(\leq)}(\mathcal{M})\), where \(\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha \text{ and } \beta \text{ are order isomorphisms among principal ideals of } (P(\leq); \{\phi_e\})\}\).

Lemma 2.4 For any \(a \in S\), \(P(Sa) (= P(Sa^*a))\) is a principal ideal of \((P(\leq); \{\phi_e\})\).

For any \(a \in S\), define a mapping \(\tau_a : \langle aa^*\rangle \to \langle a^*a\rangle\) by

\[e\tau_a = a^*ea,
\]

where \(e \in \langle aa^*\rangle\). It follows from [4] that \(\tau_a \in T_{S(\leq)}\) and \(\tau_a^* = \tau_a^*\). Moreover, for any \(a, b \in S\), \(\theta_{a^*b} = \tau_{a^*b}\). And we have the following theorem.

Theorem 2.5 Let \(S\) be a generalized inverse \(\ast\)-semigroup such that \(E(S) = E\) and \(P(S) = P\). Let \(E \sim \sum \{E_i : i \in I\}\) be the structure decomposition of \(E\) and \(P_i = P(E_i)\). Denote the restriction of the natural order on \(S\) to \(P\) by \(\leq\). For any \(e \in P\), define a mapping \(\phi_e : P \to P\) by \(f \phi_e = efe\). Then \((P(\leq); \{\phi_e\})\) is a partially ordered \(\varrho\)-set and \(T_{P(\leq)}(\mathcal{M})\) is a generalized inverse \(\ast\)-semigroup.

Moreover, for any \(a \in S\), define a mapping \(\tau_a : \langle aa^*\rangle \to \langle a^*a\rangle\) by \(e\tau_a = a^*ea\). Then a mapping \(\tau : S \to T_{P(\leq)}(\mathcal{M}) (a \mapsto \tau_a)\) is a \(\ast\)-homomorphism and the kernel of \(\tau\) is the maximum idempotent-separating \(\ast\)-congruence on \(S\).

Now, we have the following theorem.
Theorem 2.6 A generalized inverse $*$-semigroup $S$ is fundamental if and only if it is $*$-isomorphic to a $P$-full generalized inverse $*$-subsemigroup of $T_{X(\leq)}(\mathcal{M})$ on a partially ordered $\rho$-set $(X(\leq);\{\phi_x\})$ such that $P(T_{X(\leq)}(\mathcal{M}))$ is order isomorphic to $P(S)$.

Denote the sets of all partially ordered $\rho$-sets and the set of all strong $\pi$-groupoids by $P$ and $S$, respectively.

Remark 2.7 Let $(X(\leq);\{\phi_x\})$ be any element of $P$. For any $x_\rho,y_\rho \in X/\rho$ ($x_\rho \geq y_\rho$), define a mapping $\overline{\varphi}_{x_\rho,y_\rho} : X_{x_\rho} \to X_{y_\rho}$ by

$$x_\rho' \overline{\varphi}_{x_\rho,y_\rho} = y_\rho'$$

where $y_\rho' \in y_\rho$ such that $y_\rho' \leq x_\rho'$.

Moreover, we define a partial product on $X$ as follows:

$$x y = \begin{cases} x \overline{\varphi}_{x_\rho,(x_\rho)(y_\rho)} & \text{if } x \overline{\varphi}_{x_\rho,(x_\rho)(y_\rho)} = y \overline{\varphi}_{y_\rho,(x_\rho)(y_\rho)} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $(X(\leq);\{\phi_x\})\lambda = X(\pi_\rho;X/\rho;\{\overline{\varphi}_{x_\rho,y_\rho}\})$ is a strong $\pi$-groupoid, where $\pi_\rho$ is the partition of $X$ induced by $\rho$.

Conversely, let $X(\pi;Y;\{\varphi_{e,f}\})$ be any element of $S$. For any $x \in X$, define a mapping $\tilde{\phi}_x : X \to X$ by

$$y \tilde{\phi}_x = x \varphi_{e,ef},$$

where $x \in X_e$ and $y \in X_f$. If we define $\triangleright = \{(x,y) \in X \times X : x \overline{\varphi}_y = x\}$, then $X(\pi;Y;\{\varphi_{e,f}\})\mu = (X(\triangleright);\{\tilde{\phi}_x\})$ is a partially ordered $\rho$-set.

Hence the mappings $\lambda,\mu$ from $P$ to $S$ and from $S$ to $P$, respectively, are well-defined. Moreover $\mu \lambda = 1_S$, and for any $(X(\leq);\{\phi_x\}) \in P$, if $(X(\leq);\{\phi_x\})\lambda \mu = (X(\triangleright);\{\tilde{\phi}_x\})$, then $\leq = \triangleright$.

By the above argument, for any $(X(\leq);\{\phi_x\})$ in $P$, without loss of generality, we can consider $(X(\leq);\{\phi_x\})$ as a member of $P\lambda\mu$.

Now, let $X(\pi;Y;\{\varphi_{e,f}\})$ be any element of $S$. If $X(\pi;Y;\{\varphi_{e,f}\})\mu = (X(\leq);\{\phi_x\})$, then we can construct two generalized inverse $*$-semigroups $T_{X(\pi)}(\mathcal{M})$ and $T_{X(\leq)}(\mathcal{M})$. In this case, these two generalized inverse $*$-semigroups are $*$-isomorphic.
3 Extensions of $T_{X(\preceq)}(\mathcal{M})$

3.1 $T_{(X(\preceq),\sigma)}(\mathcal{M})$

By a pre-order on a set $X$ we shall mean a reflexive and transitive relation. Let $X(\preceq)$ be a pre-ordered set and let $\nu = \{(a, b) \in X \times X : a \preceq b \text{ and } b \preceq a\}$. Then $\nu$ is an equivalence relation on $X$ and $X/\nu$ is a partially ordered set with respect to the induced relation

\[(C1) \quad a\nu \subseteq b\nu \text{ if and only if } a \preceq b.\]

We call \preceq the naturally induced order on $X/\nu$ from $\preceq$. Clearly $\nu$ is the smallest equivalence relation on $X$ for which (C1) defines a partial order on $X/\nu$. We call $\nu$ the minimum partial order congruence (mpo-congruence) on $X$ from $\preceq$.

A subset $A$ of $X$ is an ideal of $X$ provided that $x \preceq y$ and $y \in A$ implies $x \in A$. For $a \in X$, we call \{x \in X : x \preceq a\} the principal ideal generated by $a$ and denote it by $\langle a \rangle$.

A bijection $\alpha$ of one pre-ordered set $X$ onto another $Y$ will be called an isomorphism provided that, for $a, b \in X$, $a \preceq b$ if and only if $a\alpha \preceq b\alpha$. In particular, if $\nu_X$ and $\nu_Y$ denote the respective mpo-congruences then $(a, b) \in \nu_X$ if and only if $(a\alpha, b\alpha) \in \nu_Y$.

Let $X(\preceq)$ be a pre-ordered set and $\nu$ the mpo-congruence from $\preceq$. Then $X$ is a partially pre-ordered $\varrho$-set if and only if $X/\nu$ is a partially ordered $\varrho$-set with respect to the naturally induced order $\preceq$ from $\preceq$.

Let $X(\preceq)$ be a partially pre-ordered $\varrho$-set and $\sigma$ an equivalence relation on $X$ such that

\[(O1) \quad \text{for any } x \text{ in } X, \langle x \rangle \text{ is an } \nu\text{-single subset with respect to } \sigma,\]

\[(O2) \quad \text{for } x, y \text{ in } X, \text{ if } (x, y) \in \sigma \text{ then } (x\nu, y\nu) \in \varrho,\]

\[(O3) \quad \text{for } x, y, z \text{ in } X, \text{ if } (x\nu)\varrho \land (y\nu)\varrho = (z\nu)\varrho, \text{ then } z_1\nu \subseteq x\nu \text{ and } z_2\nu \subseteq y\nu \text{ for some } z_1, z_2 \in (z\nu)\varrho, \text{ then for any } a \in \langle z_i \rangle, \text{ there exists } b \in \langle z_j \rangle \text{ such that } (a, b) \in \sigma, \text{ where } 1 \leq i, j \leq 2.\]

Then $(X(\preceq); \sigma)$ is called an $\omega$-set.

Let $(X(\preceq); \sigma)$ be an $\omega$-set and let $T_{(X(\preceq),\sigma)}$ denote the set of all isomorphisms from a principal ideal onto another one.

For any $\alpha, \beta \in T_{(X(\preceq),\sigma)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$\theta_{\alpha,\beta} = \{(a, b) \in r(\alpha) \times d(\beta) : (a, b) \in \sigma\}.$$
Then $\theta_{\alpha,\beta} \in T_{(X(\preceq);\sigma)}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{(X(\preceq);\sigma)}\}$, and denote a multiplication $\circ$ and a unary operation $*$ on $T_{(X(\preceq);\sigma)}$ by

$$\alpha \circ \beta = \alpha \theta_{\alpha,\beta} \beta,$$

$$\alpha^* = \alpha^{-1}.$$

Clearly, $\alpha \circ \beta$ is an isomorphism from $(z_1 \alpha^{-1})$ onto $(z_2 \beta)$. It is obvious that $T_{(X(\preceq);\sigma)}(\circ, *$) is a regular $*$-semigroup. Hence it is a generalized inverse $*$-semigroup and denoted by $T_{(X(\preceq);\sigma)}(\mathcal{M})$.

**Theorem 3.1** A regular $*$-semigroup $T_{(X(\preceq);\sigma)}(\mathcal{M})$ is a generalized inverse $*$-subsemigroup of $\mathcal{G}I_{(X(\sigma))}(\mathcal{M})$ whose set of projections is order isomorphic to $X/\nu$.

**Remark 3.2** In $T_{(X(\preceq);\sigma)}(\mathcal{M})$, if $\preceq = \subseteq$ and $\sigma = \emptyset$ then $T_{(X(\preceq);\emptyset)}(\mathcal{M}) = T_{(X(\emptyset))}(\mathcal{M})$.

Let $(X(\preceq);\sigma)$ be an $\omega$-set and let $Y = X/\nu$, where $\nu$ is the mpo-congruence from $\preceq$. For any element $\alpha$ in $T_{(X(\preceq);\sigma)}$, assume that $d(\alpha) = \langle a \rangle$. Then we can define a new mapping $\alpha' \in T_{Y(\emptyset)}$ as follows:

$$d(\alpha') = \{x\nu : x \in d(\alpha)\},$$

$$(x\nu)\alpha' = (x\alpha)\nu.$$

Then $\alpha' \in T_{Y(\emptyset)}$. Now, define a mapping $\xi : T_{(X(\preceq);\sigma)}(\mathcal{M}) \to T_{Y(\emptyset)}(\mathcal{M})$ by $\alpha\xi = \alpha'$. Then, it is easy to see that $\xi$ is a $*$-homomorphism.

**Proposition 3.3** The mapping $\xi : \alpha \mapsto \alpha'$ of $T_{(X(\preceq);\sigma)}(\mathcal{M})$ into $T_{Y(\emptyset)}(\mathcal{M})$ is a $*$-homomorphism of $T_{(X(\preceq);\sigma)}(\mathcal{M})$ onto a $P$-full generalized inverse $*$-subsemigroup of $T_{Y(\emptyset)}(\mathcal{M})$ such that $\xi \circ \xi^{-1} = \mu$, where $\mu$ is the maximum idempotent separating $*$-congruence on $T_{(X(\preceq);\sigma)}(\mathcal{M})$.

Hereafter, we shall refer to $\xi$ as the natural projection of $T_{(X(\preceq);\sigma)}(\mathcal{M})$ to $T_{Y(\emptyset)}(\mathcal{M})$.

### 3.2 Inflated representations

Let $S$ be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Define a relation $\preceq$ on $S$ by:

$$a \preceq b \text{ if and only if } a^*a \leq b^*b,$$
for $a, b \in S$. Then clearly $\preceq$ is a pre-order on $S$ for which the mpo-congruence from $\preceq$ is $\nu = \mathcal{L}$. Hence $S/\mathcal{L} = S/\nu$, under the naturally induced order $\preceq$ from $\preceq$, is just the set of $\mathcal{L}$-classes of $S$ under the usual partial ordering of the $\mathcal{L}$-classes of a generalized inverse $*$-semigroup and so is order isomorphic to the partially ordered $\rho$-set $P$ of $S$. Hence $S$ is a partially pre-ordered $\rho$-set under $\preceq$. Then $\rho = J^F|_{P}$ and hence $(av)\rho(bv) \iff a^*aJ^Eb^*b$. Hereafter, for any $a \in S$, we think $av = L_{a^*a}$ as $a^*a$.

For any $a \in S$, define a mapping $\rho_a : Sa^* \to Sa$ as follows:

$$d(\rho_a) = Sa^* (= Saa^*),$$
$$x\rho_a = xa.$$

Let $\rho : S \to GI_{(S,\Omega)}(\mathcal{M})$ by $a\rho = \rho_a$, where the relation $\Omega$ defined by: for $x, y \in S$,

$$(x, y) \in \Omega \iff x\rho_e = y \text{ for some } e \in E.$$

Since $S$ is a regular $*$-semigroup, the representation $\rho$ is faithful. Moreover, it follows from [6, Lemma 3.3] that it is a $*$-monomorphism.

**Lemma 3.4** The set $(S(\preceq); \Omega)$, defined above, is an $\omega$-set.

Again, we consider $\rho_a : Sa^* \to Sa$. By Lemma 3.4, $d(\rho_a) = \langle a^* \rangle$ and $r(\rho_a) = \langle a \rangle$. For $x, y \in d(\rho_a)$, $x^*x, y^*y \leq a^*a$. Now $x \preceq y$ if and only if $x^*x \leq y^*y$ while $xa \preceq ya$ if and only if $a^*x^*xa = (xa)^*(xa) \leq (ya)^*(ya) = a^*ya$. But, since $x^*x, y^*y \leq a^*a$ it follows that $x^*x \leq y^*y$ if and only if $a^*x^*xa \leq a^*ya$. Therefore $x \preceq y$ if and only if $xa \preceq ya$. Thus $\rho_a$ is an isomorphism of $\langle a^* \rangle$ onto $\langle a \rangle$, and hence $S\rho \subseteq T_{(S(\preceq); \Omega)}(\mathcal{M})$.

Now, we have the following theorem.

**Theorem 3.5** Let $S$ be a generalized inverse $*$-semigroup and define the relation $\preceq$ on $S$ by $a \preceq b$ if and only if $a^*a \leq b^*b$. Then $\preceq$ is a pre-order on $S$ with respect to which $S$ is a partially pre-ordered $\rho$-set, moreover $(S(\preceq); \Omega)$ is an $\omega$-set. The faithful representation $\rho$ of $S$ embeds $S$ as a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{(S(\preceq); \Omega)}(\mathcal{M})$.

If $\nu$ is the mpo-congruence on $S$ from $\preceq$, then $\nu = \mathcal{L}$ and $S/\nu$ is order isomorphic to the partially ordered $\rho$-set $P$ of $S$. Moreover, $\rho\xi = \tau$, where $\xi$ is the natural projection and $\tau$ is the representation which is defined in Theorem 2.5.

**References**


