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<td>Hirano, Norimichi</td>
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Existence of Entire Solutions for Superlinear Elliptic Problems in $R^N$

by Norimichi Hirano(Yokohama National University)

横浜国大・工 平野 載倫

1. Introduction.  In this talk, we are concerned with positive solutions of the following problem:

\[
(P) \begin{cases} 
- \Delta u + u = g(x, u), & u > 0, \quad \text{in } R^N \\
    u \in H^1(R^N), & N \geq 2
\end{cases}
\]

where $f : R^N \to R$ and $g : \Omega \times R \to R$ is continuous with $g(x, 0) = 0$ for $x \in \Omega$. In the last decade, the existence and the properties of the solutions of problem $(P)$ has been studied by many authors. Recently, the existence of positive solutions of semilinear elliptic problem

\[
(P_Q) \begin{cases} 
- \Delta u + u = Q(x) |u|^{p-1} u, & x \in R^N \\
    u \in H^1(R^N), & N \geq 2
\end{cases}
\]

has been studied by several authors, where $1 < p$ for $N = 2$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$, $Q(x)$ is positive bounded continuous function. If the function $Q(x)$ is a radial function, the existence of infinity many solutions of problem $(P_Q)$ can be shown by restricting our attention to the radial functions(cf. [1]). In case that $Q(x)$ is nonradial, we encounter a difficulty caused by lack of compact embedding of Sobolev type. In [6,7], P.L. Lions presented a method, called concentrate compactness method, which enable us to solve problems with lack of compactness, and established the following result: Assume that

\[
limit_{|x| \to \infty} Q(x) = \overline{Q}(>0) \quad \text{and} \quad Q(x) \geq \overline{Q} \quad \text{on } R^N,
\]
then problem \((P_Q)\) has a positive solution. This result is based on the observation that the ground state level \(c_Q\) of the functional

\[
I_Q(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \frac{1}{p+1} \int_{R^N} Q(x) u^{p+1} dx
\]

is lower than the ground state level \(c_{\overline{Q}}\) of functional \(I_{\overline{Q}}\). We can apply the concentrate compactness method problem \((P)\) to the problem in case that \(g : R^N \times R \to R\) satisfies \(\lim_{|x| \to \infty} g(x, t) = t^p\) and the least critical level \(c_1\) of the functional

\[
I(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \int_{R^N} \int_0^{u(x)} g(x, t) dt dx,
\]

\(u \in H^1(R^N)\), is lower than that of

\[
I^\infty(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \frac{1}{p+1} \int u^{p+1} dx.
\]

Under additional conditions on \(g\), the existence of positive solutions \((P)\) was established by Ding & Ni[4] and Stuart[10]. Recently, Cao[2] proved the existence of positive solution of \((P_Q)\) for the case that \(c_Q \leq c_{\overline{Q}}\) under the hypothesis that \(\lim_{||x|| \to \infty} Q(x) = \overline{Q}\) and \(Q(x) \geq 2^{(1-p)/2} \overline{Q}\) on \(R^N\). In case that \(c_Q = c_{\overline{Q}}\), we encounter a difficulty, because we can not apply the concentrate compactness method directly. On the other hand, in case that \(g\) is not given by the form \(Q(x)t^p\), we have to overcome another difficulty: that is, we can not use the Lagrange’s method of indeterminate coefficients. In the problem \((P_Q)\), we find a solution \(u\) of minimizing problem

\[
\inf\{I_Q(u) : u \in V_\lambda\},
\]

\[
V_\lambda = \{u \in H^1(R^N), u > 0, \int_{R^N} Q(x) u^{p+1} dx = 1\}
\]

Then \(cu\) is a solution of \((P_Q)\) for some \(c > 0\). The Lagrange’s method does not work if \(g\) is not the form \(Q(x)t^p\). Our approach enable us to treat the problem \((P)\) with \(g\) satisfying that \(g(0) = 0\) and \(g(t) \to t^p\) as \(t \to \infty\). We also consider the nonhomoginous case:

\[
(P_f)
\]

\[
\begin{align*}
-\Delta u + u &= |u|^{p-1} u + f, \quad x \in R^N \\
u &\in H^1(R^N), \quad N \geq 3
\end{align*}
\]
where $p > 1$ for $N = 1$ and $1 < p < (N + 2)/(N - 2)$ for $N \geq 3$.

The nonhomogeneous problem $(P_f)$ was studied by Zhu[12]. In [12], the existence of at least two solutions of $(P)$ was proved for nonnegative functions $f \in L^2(R^N)$ with a small $L^2$-norm and an exponential decay

\[ f(x) \leq C \exp\{- (1 + \epsilon) |x|\}, \quad \text{for } x \in R^N. \]

In the present paper, we consider multiple existence of solutions of $(P)$ for nonnegative functions $f \in L^q(R^N)$, where $q = (p + 1)/p$. Our result does not require that $f \in L^\infty(R^N)$ or any condition for the decay of $f$ at infinity.

In this talk, we show an approach for problems $(P)$ and $(P_f)$ based on arguments using singular homology theory. Throughout this paper, we denote by $| \cdot |_q$ the norm of $L^q(R^N)$. We impose the following conditions on the continuous mapping $g : R^N \times R \rightarrow R$:

\begin{enumerate}[(g1)]
  \item There exists a positive number $d < 1$ such that
    
    \[ -dt + (1 - d)t^p \leq g(x, t) \leq dt + (1 + d)t^p \]
    
    for all $(x, t) \in R^N \times [0, \infty)$;
  
  \item there exists a positive number $C$ such that
    
    \[ |g_t(x, 0)| < 1 \text{ and } 0 < t^2 g_{tt}(x, t) < C(1 + t^p) \]
    
    for all $(x, t) \in R^N \times [0, \infty)$;
  
  \item
    \[ \lim_{|x| \rightarrow \infty} g(x, t) = |t|^{p-1} t \]
    
    uniformly on bounded intervals in $[0, \infty)$,
\end{enumerate}

where $1 < p$ for $N = 2$ and $1 < p < (N + 2)/(N - 2)$ for $N \geq 3$, and $g_t(\cdot, \cdot)$ stands for the derivative of $g$ with respect to the second variable.

We can now state our main results.

**Theorem 1.** Suppose that (g2) and (g3) holds. Then there exists $d_0 > 0$ such that if (g1) holds with $d < d_0$, then problem $(P)$ has a positive solution.

For problem $(P_f)$, we have

**Theorem 2.** There exists a positive number $C$ such that for each $f \in L^q(R^N)$, with $f \geq 0$ and $|f|_q < C$, problem $(P_f)$ possesses at least two solutions.
2. Preliminaries. We just give a sketch of a proof of Theorem 1 to show that how the singular homology theory works for the proof of existence of positive solutions. We put $H = H^1(R^N)$. Then $H$ is a Hilbert space with norm 

$$\|u\| = \left(\int_{R^N} (|\nabla u|^2 + |u|^2) dx\right)^{1/2}.$$ 

The norm of the dual space $H^{-1}(R^N)$ of $H$ is also denoted by $\|\cdot\|$. $B_r$ stands for the open ball centered at 0 with radius $r$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $H^1(R^N)$ and $H^{-1}(R^N)$. For each $r > 1$, the norm of $L^r(R^N)$ is denoted by $|\cdot|_r$. For simplicity, we write $|\cdot|_*$ instead of $|\cdot|_{p+1}$. For $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$. We denote by $C_p$ the minimal constant satisfying 

$$|u|_* \leq C_p \|u\| \quad \text{for } u \in H. \quad (2.1)$$ 

It is easy to check that critical points of $I$ are solutions of (P). It is also obvious that nonzero critical points of $I^\infty$ are solutions of (P) with $g(t) = t^p$ for $t \geq 0$. For each functional $F$ on $H$ and $a \in R$, we set $F_a = \{u \in H : F(u) \leq a\}$. We put 

$$M = \{u \in H \backslash \{0\} : \|u\|^2 = \int_{R^N} u g(x, u) dx\}$$ 

$$M^\infty = \{u \in H \backslash \{0\} : \|u\|^2 = \int_{R^N} u^{p+1} dx\}$$ 

For the proof of the following two propositions are crucial:

**Proposition 2.1.** There exists positive number $d_0 < \tilde{d}_0$ and $\epsilon_0$ satisfying that if $(g1)$ holds with $d \leq d_0$, then for each $0 < \epsilon < \epsilon_0$, 

$$H_*(I^\infty_{c+\epsilon}, I^\infty_\epsilon) = H_*(I^\infty_{c+\epsilon}, I^\infty_\epsilon)$$ 

where $H_*(A, B)$ denotes the singular homology group for a pair $(A, B)$ of topological spaces(cf. Spanier[8]).

**Proposition 2.2.** For each positive number $\epsilon < \epsilon_0$, 

$$H_q(I^\infty_{c+\epsilon}, I^\infty_\epsilon) = \begin{cases} 2 & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$
Here we give a proof for Proposition 2.2.

We set
\[ T_{u_{\infty}}(M^{\infty}) = \{ \lim_{t \to 0} (c(t) - u_{\infty})/t : c \in C^{1}((-1,1); M^{\infty}) \text{ with } c(0) = u_{\infty} \}, \]
\[ C = C_{-} \cup C_{+} = \{ -\tau_{x}u_{\infty} : x \in \mathbb{R}^{N} \} \cup \{ \tau_{x}u_{\infty} : x \in \mathbb{R}^{N} \} \]
and
\[ T_{u_{\infty}}(C) = \{ \lim_{t \to 0} (u_{\infty}(\cdot + tx) - u_{\infty}(\cdot))/t : x \in \mathbb{R}^{N} \}. \]
It follows from the definition of \( f^{\infty} \) that the codimension of \( T_{u_{\infty}}(M^{\infty}) \) in \( H \) is one. It is also obvious that \( \dim T_{u_{\infty}}(C) = N \). We denote by \( \bar{H} \) the subspace such that \( H = \bar{H} \oplus T_{u_{\infty}}(C) \). For each \( r > 0 \), we set \( B_{r}^{0} = B_{r} \cap H \).

Here we consider the linealized equation
\[ (L) \quad -\Delta u + u - h(x)u = \mu u, \quad u \in H, \mu \in \mathbb{R}, \]
where \( h(x) = p \left| u_{\infty}(x) \right|^{p-1} \) for \( x \in \mathbb{R}^{N} \). Since \( -\Delta \) is positive definite and \( h(x)I \) is compact, we find by Freidrich's theory that the negative spectrums of \( A = -\Delta - h(x)I \) are finite and each eigenspace corresponding to a negative eigenvalue is finite dimensional. Then each eigenspace corresponding to a nonpositive eigenvalue of \( L = -\Delta + I - h(x)I \) is finite dimensional. Then there exists \( c_{0} > 0 \) and a decomposition \( H = H_{-} \oplus H_{0} \oplus H_{+} \) such that \( H_{0} = \ker(L) \) and \( L \) is positive(negative) definite on \( H_{+}(H_{-}) \) with
\[ \langle Lv, v \rangle \geq c_{0} \| v \|^{2} \quad (\leq -c_{0} \| v \|^{2}) \quad \text{for } v \in H_{+}(H_{-}). \]

Since each \( u \in C \) is a solution of problem \((P_{\infty})\), we can see that \( T_{u_{\infty}}(C) \subset H_{0} \).

**Lemma 2.3.** \( \dim H_{-} = 1 \).

**Proof.** Since \( I^{\infty} \) attains its minimal on \( M^{\infty} \) at \( u_{\infty} \), we have that \( T_{u_{\infty}}(M^{\infty}) \subset H_{+} \oplus H_{0} \). Then since the codimension of \( M^{\infty} \) is one, we find that \( \dim H_{-} \leq 1 \). On the other hand, we have
\[ \langle Lu_{\infty}, u_{\infty} \rangle = \int_{\mathbb{R}^{N}} (|\nabla u_{\infty}|^{2} + |u_{\infty}|^{2} - p |u_{\infty}|^{p+1})dx \]
\[ < \int_{\mathbb{R}^{N}} (|\nabla u_{\infty}|^{2} + |u_{\infty}|^{2} - |u_{\infty}|^{p+1})dx = 0. \]
Then we have that $\dim H_- \geq 1$. This completes the proof.

In the following we denote by $\varphi$ an element of $H_-$ with $\|\varphi\| = 1$. Here we note that since $h \in C^\infty(R^n)$, each solution $u$ of (L) is in $C^1(R^n)$. It then follows that if $u$ has the form

$$u(r, \theta) = \psi(r)\xi(\theta_1, \cdots, \theta_{n-1}), \quad \text{with } \xi \neq \text{const.},$$

in spherical coordinate, $\psi$ satisfies that $\psi(0) = 0$.

We denote by $H_r$ the set of all radial functions in $H$ and by $(L_r)$ the problem (L) restricted to $H_r$. Then, in spherical coordinates, the problem $(L_r)$ with $\mu > 0$ is reduced to

$$\psi''(r) + \frac{n-1}{r} \psi'(r) + (h-1)\psi = -\mu\psi(r), \quad r > 0, \psi \in C_r, \quad (2.3)$$

$$\frac{d\psi(r)}{dr}(0) = 0, \quad (2.4)$$

where $C_r = \{\psi \in C[0, \infty) : \lim_{r \to \infty} \psi(r) = 0\}$.

We next consider nonradial solutions of (L). In case of nonradial functions, the problem (L) is deduced to

$$\psi''(r) + \frac{n-1}{r} \psi'(r) + ((h-1) - \frac{\alpha_k}{r^2})\psi(r) = -\mu\psi(r), \quad r > 0, \psi \in \mathcal{H}_r$$

$$\psi(0) = 0 (2.6)$$

where $\alpha_k = k(k+n-1)$, $k = 1, 2, \cdots$. Note that $\alpha_k$ are the eigenvalues of Laplacian $-\triangle$ on $S^{n-1}$, the unit sphere, and the dimension of the eigenspace $S_k$ associate with $\alpha_k$ is

$$\rho_k = \binom{k+n-2}{k} \frac{n+2k-2}{n+k-2}.$$ 

That is there exists smooth functions $\{\varphi_{k,i} : i = 1, \cdots, \rho_k\}$ defined on $S^{n-1}$ such that $S_k = \text{span}\{\varphi_{k,1}, \cdots, \varphi_{k,\rho_k}\}$, and the functions $u = \psi(r)\varphi_{k,i}(\theta)$ are the solutions of (L).

**Lemma 2.4.** $\dim H_0 \leq N + 1$.

**Proof.** Since $\dim H_- = 1$ and $u_\infty \in H_r$, we have by (2.2) that the problems (2.3), (2.4) has exactly one negative eigenvalue. We also note
that each nonpositive eigenvalue $\mu$ of problems (2.3), (2.4) is simple. Then the dimension of $H_{0,r} = H_{0} \cap H_{r}$ is at most one.

We next consider nonradial cases. That is we will see that the eigenspace of the problem (2.5) with $\mu = 0$ is N-dimensional space. Recalling that $\nabla I(v) = 0$ on $C$, we can see that

$$-\Delta v + v - h(x)v = 0 \quad \text{for all } v \in T_{u_\infty}(C). \quad (2.7)$$

That is $T_{u_\infty}(C) \subset H_0$. Since $\dim T_{u_\infty}(C) = N$, we have that $\dim H_0 \geq N$. On the other hand, since $u_\infty$ satisfies

$$u''(r) + \frac{n-1}{r}u'(r) + p |u_\infty|^{p-1} u(r) = 0, \quad (2.8)$$

we find that $v(r) = u'_\infty$ satisfies

$$v''(r) + \frac{n-1}{r}v'(r) + ((h(x) - 1) - \frac{\alpha_1}{r^2})v(r) = 0.$$

Then we find that the N-dimensional space $\overline{C} = \text{span}\{v(r)\varphi_{1,i} : i = 1, \cdots, n-1\}$ is a subspace of solution set of (L) with $\mu = 0$. We claim that there exists no nonradial solution of (L) with $\mu = 0$ which is not contained in $\overline{C}$. Suppose contrary, there exists a nonradial solution $z$ of (L) with $\mu = 0$ such that $z \perp \overline{C}$. Then there exists $\psi \in C_r$ such that

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h(x) - 1) - \frac{\alpha_k}{r^2})\psi(r) = 0$$

for some $k > 1$ and $z = \psi(r)\varphi_{k,i}$ are solutions of (L) with $\mu = 0$. The equality above can be rewritten as

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h(x) - 1) - \frac{\alpha_k - \alpha_1}{r^2})\psi(r) - \frac{\alpha_1}{r^2}\psi(r) = 0.$$

Then $u = \psi(r)\varphi_{1,1}$ is a solution of problem

$$-\Delta u + u - h(x)u = \frac{\alpha_1 - \alpha_k}{r^2}u.$$

It then follows that

$$< -\Delta u + u - h(x)u, u >= 0. \quad (2.9)$$
Since $u$ is orthogonal to $\varphi$, we obtain from (2.9) that $\dim H_- \geq 2$. This is a contradiction. Thus we obtain that $H_0 = T_{u_0}(C) \oplus H_{0,r}$ and then $\dim H_0 \leq N + 1$.

Here we recall that $H$ has a decomposition $H = \tilde{H} \oplus T_{u_\infty}(C)$ and then $H = \tau_x \tilde{H} \oplus \tau_x T_{u_\infty}(C)$ for each $x \in R^N$. Then since $C_\pm$ are smooth $N$–manifolds, we have that there exists $r_0 > 0$ such that

$$\tau_x((-1)^i u_\infty + B_{r_0}) \cap \tau_y(u_\infty + B_{r_0}) = \phi$$

(2.10) for all $x, y \in R^N$ with $x \neq y$, and $i = 0, 1$. Here we consider a restriction $I^\infty |_{u_\infty + \tilde{H}}$ of $I^\infty$ on $u_\infty + \tilde{H}$. Then from Lemma 3.2 and Lemma 3.3, we have by Gromoll-Meyer theory[3] that there exists subspaces $H_1$, $H_{2,1}$, $H_{2,2}$ of $\tilde{H}$, a positive number $r_1 < r_0$, a mapping $\beta \in C^1((H_{2,2} \cap B_{r_1}^0), R)$ and a homeomorphism $\psi : u_\infty + B_{r_1}^0 \to u_\infty + \tilde{H}$ such that $\tilde{H} = H_1 \oplus H_{2,1} \oplus H_{2,2}$ and

$$I^\infty |_{u_\infty + \tilde{H}} (\psi (u)) = c - \| u_1 \|^2 + \| u_{2,1} \|^2 + \beta(u_{2,2})$$

(2.11) for each $u \in u_\infty + B_{r_1}^0$ with $u = u_\infty + u_1 + u_{2,1} + u_{2,2}$, $u_1 \in H_1$, $u_{2,i} \in H_{2,i}$, $i = 1, 2$. It follows from Lemma 2.3 that $H_{2,2}$ is one dimensional. Noting that $T_{u_\infty}(M) \subset H_0 \oplus H_+$ and $u_\infty$ is the minimal point of $I^\infty$ on $M$, we have by choosing $r_1$ sufficiently small that $\beta(t \varphi_2)$ is strictly increasing as $| t |$ increases in $[-r_1, r_1]$, where $\varphi_2 \in H_{2,2}$ with $\| \varphi_2 \| = 1$.

Since $I^\infty$ is even, it is obvious that $I^\infty$ has the form (2.11) on $-(u_\infty + B_{r_1}^0)$. We also note that for each $x \in R^N$, (2.11) holds for each $u \in \tau_x(u_\infty + B_{r_0}^0)$ with $\psi$ replaced by $\tau_{-x} \circ \psi$.

**Proof of Proposition 2.2.** By the deformation property(cf. theorem 1.2 of Chang[3]) and the homotopy invariance of the homology groups, we have

$$H_q(I_c^\infty \setminus C, I_c^\infty) \cong H_q(I_c^\infty, I_c^\infty), \ \text{and}$$

$$H_q(I_c^\infty \setminus C, I_c^\infty) \cong H_q(I_c^\infty, I_c^\infty) \cong 0.$$

From the exactness of the singular homology groups,

$$H_q(I_c^\infty \setminus C, I_c^\infty) \to H_q(I_c^\infty, I_c^\infty) \to H_q(I_c^\infty, I_c^\infty \setminus C)$$

$$\to H_{q-1}(I_c^\infty \setminus C, I_c^\infty) \to \cdots$$
we find
\[ 0 \to H_q(I^\infty_c, I^\infty_{c-\epsilon}) \to H_q(I^\infty_c, I^\infty_c \setminus C) \to 0. \]
That is
\[ H_q(I^\infty_c, I^\infty_{c-\epsilon}) \cong H_q(I^\infty_c, I^\infty_c \setminus C). \]
Noting that \( \bigcup \{ \tau_x(\pm u_\infty + B_{r_1}^0) : x \in R^N \} \) are disjoint open neighborhoods of \( C_\pm \) respectively, and that \( I^\infty \) is invariant under the translations \( \tau_x \), we find from the excision property and (2.11) that
\[
\begin{align*}
H_*(I^\infty_{c+\epsilon}, I^\infty_{\epsilon}) \\
\cong H_*(I^\infty_c, I^\infty_c \setminus C) \\
\cong H_*(I^\infty_c \cap (\bigcup_{i=\pm 1} \bigcup_{x} \tau_x(iu_\infty + B_{r_1}^0) \setminus \{u_\infty\})) \\
\cong H_*(u_\infty + B_{r_1}^1, (u_\infty + B_{r_1}^1) \setminus \{u_\infty\}) \\
\oplus H_*(-u_\infty + B_{r_1}^1, (-u_\infty + B_{r_1}^1) \setminus \{u_\infty\}) \\
\cong H_*([0,1], \{0,1\}) \oplus H_*([0,1], \{0,1\}).
\end{align*}
\]
This completes the proof.

3. Proof of Theorem 1. We next consider a triple \((U,K,\epsilon) \subset H \times H \times R^+\) satisfying the following conditions:

(1) \( U \cap (-U) = \phi \);
(2) \( \{ \tau_x u_\infty : |x| \geq r \} \subset int K \) for some \( r > 0 \);
(3) \( \text{cl}(I_{c+\epsilon} \cap K) \subset int(I_{c+\epsilon} \cap U) \);
(4) \( H_{N-1}(I_{c+\epsilon} \cap U) = 1, \ H_1(I_{c+\epsilon} \cap U) = 0; \)
(5) \( I_{\epsilon} \) is a strong deformation retract of \( I_{c+\epsilon} \setminus (K \cup (-K)) \);
(6) \( H_{N-1}((I_{c+\epsilon} \cap U) \setminus K) = 2 \) or \( H_0((I_{c+\epsilon} \cap U) \setminus K) \geq 2 \) holds.

Proposition 3.1. There exists a triple \((U,K,\epsilon) \subset H \times H \times R^+\) which satisfies (1) - (6).

We omit the proof of Proposition 3.1.
Lemma 3.2. Suppose that there exist a triple \((U, K, \epsilon) \subset H \times H \times R^+\) satisfying (1)-(6). Suppose in addition that \(H_{N-1}((I_{c+\epsilon} \cap U) \setminus K) \geq 2\). Then \(H_N(I_{c+\epsilon}, I_\epsilon) \geq 2\).

Proof. We put \(\tilde{K} = K \cup (-K)\). Since \(I_\epsilon\) is a strong deformation retract of \(I_{c+\epsilon} \setminus \tilde{K}\), we find that

\[
H_q(I_{c+\epsilon} \setminus \tilde{K}, I_\epsilon) \cong H_q(I_\epsilon, I_\epsilon) \cong 0.
\]

Then we have from the exactness of the singular homology groups of the triple \((I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}, I_\epsilon)\) that

\[
0 \rightarrow H_q(I_{c+\epsilon}, I_\epsilon) \rightarrow H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}) \rightarrow 0.
\]

That is

\[
H_q(I_{c+\epsilon}, I_\epsilon) \cong H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}).
\]

From (1), we find

\[
H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}) \cong H_q(W, W \setminus K) \oplus H_q(-W, (-W) \setminus (-K))
\]

where \(W = I_{c+\epsilon} \cap U\). Then since \(H_{N-1}(W \setminus K) \geq 2\), we have from (4) and the exactness of the sequence

\[
\cdots \rightarrow H_q(W, W \setminus K) \rightarrow H_{q-1}(W \setminus K) \rightarrow H_{q-1}(W) \rightarrow H_{q-1}(W, W \setminus K) \rightarrow \cdots
\]

with \(q = N\) that \(H_N(I_{c+\epsilon}, I_\epsilon) \cong H_N(W, W \setminus K) \oplus H_N(W, W \setminus K) \geq 2\).

Lemma 3.3. Suppose that \((U, K, \epsilon) \subset H \times H \times R^+\) satisfies (1) - (6). Suppose in addition that \(H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \setminus K) = 1\). Then \(H_1(I_{c+\epsilon}, I_\epsilon) = 0\) or \(H_0(I_{c+\epsilon}, I_\epsilon) = 2\) holds.

Proof. From the argument in the proof of Proposition 3.2, we have that

\[
H_1(I_{c+\epsilon}, I_\epsilon) \cong H_1(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U) \setminus K) \oplus H_N(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U) \setminus K).
\]

Then since \(H_1(I_{c+\epsilon} \cap U) = 0\), and \(H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \setminus K) = 1\), the assertion follows from the exactness of the sequence (3.1) with \(q = 1\).
We can now prove Theorem 1.

**Proof of Theorem.** Let \((U, K, \epsilon)\) be the triple constructed above. We have by Proposition 2.1 and Proposition 2.2 that \(H_1(I_{c+\epsilon}, I_\epsilon) = 2\) and \(H_q(I_{c+\epsilon}, I_\epsilon) = 0\) for \(q \neq 1\). Now suppose that \((I_{c+\epsilon} \cap U)\setminus K\) is disconnected. Then since
\[H_0((I_{c+\epsilon} \cap U)\setminus K) \geq 2,\]
we find by Lemma 3.2 that \(H_N(I_{C+\epsilon}, I_\epsilon) = 2\). This is a contradiction. On the other hand, if \(U\setminus K\) is connected, then \(H_0(U\setminus K) = 1\). Then by Lemma 3.3, we have \(H_1(I_{c+\epsilon}, I_\epsilon) = 0\) or \(H_0(I_{c+\epsilon}, I_\epsilon) = 2\). This is a contradiction. Thus we obtain that there exists a positive solution of \((P)\).

**REFERENCES**


