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Truth-table reductions and minimum sizes of forcing conditions (preliminary draft)

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Abstract

This note is a refinement of our former note [KSY05] "Logarithmic truth-table reductions and minimum sizes of forcing conditions (preliminary draft)" ***Sūrikaiseki-kenkyūsho Kōkyūroku*** 1442 (2005), 42-47. The current note extends and corrects [KSY05]. In our former works, for a given concept of reduction, we study the following hypothesis: "For a random oracle A, with probability one, the degree of the one-query tautologies with respect to A is strictly higher than the degree of A." In our former works, the following three results

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are shown: (1) the hypothesis for polynomial-time Turing reduction is equivalent to the assertion that the probabilistic complexity class R is not equal to NP, (2) the hypothesis for polynomial-time truth-table reduction implies that P is not NP, (3) [KSY05] the hypothesis holds for $(\log n)^{O(1)}$-question truth-table-reduction (without polynomial-time bound). In this note, we show that if ε is an enough small positive number, then we can substitute $\varepsilon\ell$ for $(\log n)^{O(1)}$ in the statement of (3), where ℓ denotes the total number of occurrences of symbols in a relativized formula. We also show the hypothesis holds for monotone truth-table reduction.

1 Preface

In our former works [Su98, Su99, Su00, Su01, Su02, Su05, KSY05], by extending the work of Ambos-Spies [Am86] and related works, we consider the relationships with the canonical product measure of Cantor space and complexity of one-query tautologies. A formula F of the relativized propositional calculus is called *a one-query forumla* if F has exactly one occurrence of a query symbol. For example,

$$(q_0 \Leftrightarrow \xi^3(q_1, q_2, q_3)) \Rightarrow (q_1 \Rightarrow q_0)$$

is a one-query formula, where q_0, q_1, q_2, q_3 are usual propositional variables. We assume that each propositional variable takes the value 0 or 1 (0 denotes false and 1 denotes true). And, ξ^3 in the above formula is a query symbol. For a given oracle A, a function A^3 is defined as follows, where λ is the empty string, and the query symbol ξ^3 is interpreted as the function A^3.

$$\begin{array}{llll} A^3(000) = A(\lambda), & A^3(001) = A(0), & A^3(010) = A(1), & A^3(011) = A(00), \\ A^3(100) = A(01), & A^3(101) = A(10), & A^3(110) = A(11), & A^3(111) = A(000). \end{array}$$

Thus, more informally, the following holds for each $j = 0, 1, \cdots, 2^3 - 1$, where the order of strings is defined as the canonical length-lexicographic order.

$$A^3(\text{ the } (j+1)\text{st 3-bit string }) = A(\text{ the } (j+1)\text{st string }).$$

For each n, an n-ary Boolean function A^n is defined in the same way, and an interpretation of the query symbol ξ^n is defined in the same way. For an oracle A, the concept of a *tautology with respect to A* is defined in a natural way. If a one-query formula F is a tautology with respect to A, then we say F is a *one-query tautology with respect to A*. The set of all one-query tautologies with respect to A is denoted by $1\mathrm{TAUT}^A$.

In [Su02], for a given concept $\leq_\alpha$ of reduction, we study the following hypothesis, where $1\mathrm{TAUT}^X$ denotes the set of all one-query tautologies with respect to an oracle X.

One-query-jump hypothesis for $\leq_\alpha$: The class $\{X : 1\mathrm{TAUT}^X \leq_\alpha X\}$ has measure zero.

For a given reduction $\leq_\alpha$, we denote the corresponding one-query-jump hypothesis by $[\leq_\alpha]$.

In [Su98], it is shown that the one query-jump hypothesis for p-T reduction is equivalent to "R $\neq$ NP."

And, in [Su02], it is shown that the one query-jump hypothesis for p-tt reduction implies "P $\neq$ NP."

In [Su05], we show that the one query-jump hypothesis for p-btt reduction holds, where p-btt denotes polynomial-time bounded-truth-table reduction. The anonymous referee of [Su05] noticed that the one query-jump hypothesis holds for bounded-truth-table reduction without polynomial-time bound, and Kumabe independently noticed the same result. The referee's proof, which may be found in [Su05], uses some concepts of resource-bounded generic oracles in [AM97]. Kumabe's proof is more simple.

In [KSY05] we show that the one query-jump hypothesis holds for $(\log n)^{O(1)}$-question tt-reduction (without polynomial-time bound).

A Boolean formula is called *monotone* if every propositonal conncetive in it is either disjunction or conjunction, and it does not have an occurrences of negation symbol. A tt-reduction is called a *monotone tt-reduction* if its truth table is monotone for every input. In §3, we show that the one query-jump hypothesis holds for monotone tt-reduction (without polynomial-time bound). In §4, we show the following. If ε is an enough small positive number then the one query-jump hypothesis holds for $\varepsilon\ell$-question tt-reduction (without polynomial-time bound), where ℓ denotes the total number of occurrences of symbols in a relativized formula. In §5, we apply the result of §4 to minimum sizes of forcing conditions.

Corrigendum to our former note Theorem 4 in our former note [KSY05, p.45] has an error in its proof.

2 Notation

Most of our notation is the same as that of [Su02], [Su05] and [KSY05]. Almost all undefined notions may be found in these papers.

ω stands for $\{0, 1, 2, 3\cdots\}$, while $\mathbb{N}$ stands for $\{1, 2, 3\cdots\}$. In some textbooks, the complexity class R is denoted by RP. For the detail of the class R, see for example [BDG88].

The definition of polynomial-time truth-table reduction and its variant may be found in [LLS75].

monotone tt-reduction

If A is tt-reducible to B via f and, if for any input x, propositional connectives used in the truth table (i.e., the φ_x of $f(x) = (\varphi_x, s_{x,1}, \cdots, s_{x,k})$) is conjunction and disjunction only, and negation is not used, then we say "A is monotone tt-reducible to B via f". If A is monotone tt-reducible to B via some function, then we say "A *is monotone tt-reducible to* B".

$\ell(F)$, **length of a formula**

In this note, a given relativised formula F, the symbol $\ell(F)$ denotes the total number of occurrences of propositional variables (q_0, q_1, q_2, $\cdots$), propositonal connectives ($\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$), query symbols (ξ^1, ξ^2, ξ^3, $\cdots$) and punctuation marks (commas, parentheses). In the case of a given string x is not (the binary code of) a relativized formula, the symbol $\ell(x)$ denotes the binary length of x.

$\varepsilon\ell$**-question tt-reduction**

Suppose that ε is a positive real number. If A is tt-reducible to B via f and, if for any input x it holds that

$$k \leq \varepsilon\ell(x),$$

where k is the norm of f at x, then we say "A is $\varepsilon\ell$-question tt-reducible to B via f". If A is $\varepsilon\ell$-question tt-reducible to B via some function, then we say "A *is* $\varepsilon\ell$*-question tt-reducible to* B".

3 Monotone truth table redcution

Theorem 1 *The Lebesgue measure of the set*

$$\{X : 1\mathrm{TAUT}^X \textit{ is monotone tt-reducible to } X\}$$

is zero. In other words, one-query jump hypothesis holds for monotone tt-reduction (without polynomial-time bound).

4 The case where norm is linear of length of a formula

Theorem 2 (*Main Theorem*) *Let ε be a positive real number and suppose that ε is enough small. Then the Lebesgue measure of the following class is zero.*

$$\{X : 1\mathrm{TAUT}^X \leq_{\varepsilon\ell\text{-tt}} X\}$$

In other words, the one-query-jump hypothesis holds for $\varepsilon\ell$-question tt-reduction (without polynomial-time bound).

5 Lower bounds for forcing complexity

Theorem 3 *Let ε be a positive real number and suppose that ε is enough small. Let $\mathcal{D}_{\varepsilon\ell}$ be the class of all oracles D such that there exists a positive integer c (c may* *depend on D) of the following property. For any $F \in 1\mathrm{TAUT}^D$ such that $\ell(F) \geq c$, there exists a forcing condition S such that S is a subfunction of D, S forces F to be a tautology and such that $|\operatorname{dom} S| \leq \varepsilon\ell(F)$, where the left-hand side denotes the cardinality of* $\operatorname{dom} S$. *Then $\mathcal{D}_{\varepsilon\ell}$ has measure zero.*

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