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Kyoto University
Borel classes dimensions

1 Introduction and results.

The classes of topological spaces are assumed to be

1. non-empty (we suppose that at least the empty space $\emptyset$ is a member), and
2. monotone with respect to closed subsets.

The letter $\mathcal{P}$ is used to denote a such class and the following classes of spaces satisfy the conditions 1 and 2 above.

- The class of compact metrizable spaces $\mathcal{K}$.
- The class of $\sigma$-compact metrizable spaces $\mathcal{S}$.
- The class of completely metrizable spaces $\mathcal{C}$.
- The class of separable completely metrizable spaces $\mathcal{C}_0$. 
Let $X$ be a space and $A$, $B$ disjoint subsets of $X$. We recall that a closed set $C \subset X$ is said to be a partition between $A$ and $B$ in $X$ if there are disjoint open subsets $U$ and $V$ of $X$ such that $A \subset U$, $B \subset V$ and $C = X \setminus (U \cup V)$.

In [4] Lelek introduced the small inductive dimension modulo a class $\mathcal{P}$, $\mathcal{P}$-ind, which is a natural generalization of well known dimension functions such as the small inductive dimension $\ind$ and the small inductive compactness degree $\cmp$.

**Definition 1.1** Let $X$ be a regular $T_1$-space and $\mathcal{P}$ a class of spaces. Then we define the small inductive dimension modulo a class $\mathcal{P}$, $\mathcal{P}$-ind $X$, of $X$ as follows.

(i) $\mathcal{P}$-ind $X = -1$ iff $X \in \mathcal{P}$.

(ii) For a natural number $n$, $\mathcal{P}$-ind $X \leq n$ if for any point $x \in X$ and any closed subset $A$ of $X$ with $x \notin A$ there exists a partition $C$ between $x$ and $A$ in $X$ such that $\mathcal{P}$-ind $C < n$.

The small inductive dimension modulo a class $\mathcal{P}$ has a natural transfinite extension.

**Definition 1.2** Let $X$ be a regular $T_1$-space and $\alpha$ either an ordinal number or the integer $-1$. Then the small transfinite inductive dimension modulo $\mathcal{P}$, $\mathcal{P}$-trind $X$, of $X$ is defined as follows.

(i) $\mathcal{P}$-trind $X = -1$ iff $X \in \mathcal{P}$;

(ii) $\mathcal{P}$-trind $X \leq \alpha$ if for any point $x \in X$ and any closed subset $A$ of $X$ with $x \notin A$ there exists a partition $C$ between $x$ and $A$ in $X$ such that $\mathcal{P}$-trind $C < \alpha$.

(iii) $\mathcal{P}$-trind $X = \alpha$ if $\mathcal{P}$-trind $X \leq \alpha$ and $\mathcal{P}$-trind $X > \beta$ for any ordinal $\beta < \alpha$;

(iv) $\mathcal{P}$-trind $X = \infty$ if $\mathcal{P}$-trind $X > \alpha$ for any ordinal $\alpha$.

We notice the following.

- $\emptyset$-trind $X = \trind X$, i.e., the small transfinite dimension.
• \(\kappa\)-ind \(X = \text{cmp} X\) (and \(\kappa\)-trind \(X = \text{trcmp} X\)), i.e., the small (transfinite) compactness degree.

• \(C\)-ind \(X = \text{id} X\) (and \(C\)-trind \(X = \text{trid} X\)), i.e., the small (transfinite) completeness degree.

• If \(P_2 \subset P_1\), then \(P_1\)-trind \(X \leq P_2\)-trind \(X\); in particular, \(\text{id} X \leq \text{trcmp} X \leq \text{trid} X\) holds.

Here, we shall consider on the absolute Borel classes. For each ordinal number \(\alpha\), let \(A(\alpha)\) and \(M(\alpha)\) be the absolute additive class \(\alpha\) and the absolute multiplicative classe \(\alpha\), respectively. Further, \(A(\alpha) \cap M(\alpha)\) is said to be the absolute ambiguous class \(\alpha\) and we write \(AB = \cup\{A_\alpha : \alpha < \omega_1\}\).

We notice that the absolute Borel classes in the universe of metrizable spaces satisfy the conditions 1 and 2.

Recall that in the universe of separable metrizable spaces, we have the following.

• \(A(0) = \{\emptyset\}\).

• \(M(0) = \kappa\).

• \(A(1) = S\).

• \(M(1) = C_0\).

• A diagram of the hierarchy of absolute Borel classes:

\[
\begin{align*}
\{\emptyset\} & \subseteq \mathcal{K} \subseteq A(1) \cap M(1) \\
A(1) & = S \\
& \subseteq A(2) \cap M(2) \\
M(1) & = C_0 \\
& \subseteq M(2)
\end{align*}
\]

We have a trivial example which shows the difference between trind and trcmp: The Hilbert cube \(\mathbb{I}\) has trind \(\mathbb{I}\) = \(\infty\) and cmp \(\mathbb{I}\) (\(= \text{id} \mathbb{I}\) = \(S\)-ind \(\mathbb{I}\)) = \(-1\). Furthermore, E. Pol constructed the following example.
Example 1.1 (E. Pol, [5]) There exists a $\sigma$-compact, completely metrizable space $P$ such that trcmp $P = \infty$ (i.e., trind $P = \text{trcmp} P = \infty$ and tricd $P = S\text{-trind } P = A(1) \cap M(1)\text{-trind } P = -1$).

Thus, we may ask whether we can generalize Pol's example to every ordinal number $\alpha < \omega_1$.

It is well known that the small compactness degree cmp is related to an extension property, i.e., de Groot proved that a separable metrizable space $X$ is rim-compact (i.e., cmp $X \leq 0$) iff $X$ has a metric compactification $Y$ such that dim($Y \setminus X$) $\leq 0$. Connect with this theorem, we introduce other two dimension-like functions.

Definition 1.3 Let $\mathcal{P}$ be a class of spaces. We recall that a separable metrizable space $Y$ is a $\mathcal{P}$-hull (resp. $\mathcal{P}$-kernel) of a separable metrizable space $X$ if $Y \in \mathcal{P}$ and $X \subset Y$ (resp. $Y \subset X$). Then the small transfinite $\mathcal{P}$-deficiency, $\mathcal{P}$-trdef $X$, and the small transfinite $\mathcal{P}$-surplus, $\mathcal{P}$-trsur $X$, of a separable metrizable space $X$ are defined by

$$\mathcal{P}\text{-trdef } X = \min\{\text{trind } (Y \setminus X) : Y \text{ is an } \mathcal{P}\text{-hull of } X\},$$

$$(\mathcal{P}\text{-def } X = \min\{\text{ind } (Y \setminus X) : Y \text{ is an } \mathcal{P}\text{-hull of } X\}),$$

$$\mathcal{P}\text{-trsur } X = \min\{\text{trind } (X \setminus Y) : Y \text{ is an } \mathcal{P}\text{-kernel of } X\},$$

$$(\mathcal{P}\text{-sur } X = \min\{\text{ind } (X \setminus Y) : Y \text{ is an } \mathcal{P}\text{-kernel of } X\}).$$

It is clear that the functions $\mathcal{P}$-trdef and $\mathcal{P}$-trsur are transfinite extensions of the functions $\mathcal{P}$-def and $\mathcal{P}$-sur, respectively, which are discussed in [1]. It is also clear that if $\mathcal{P}_2 \subset \mathcal{P}_1$, then $\mathcal{P}_1\text{-trdef } X \leq \mathcal{P}_2\text{-trdef } X$ and $\mathcal{P}_1\text{-trsur } X \leq \mathcal{P}_2\text{-trsur } X$.

Recall also that for the function $\mathcal{K}$-def is the well known compact deficiency def. We will denote the transfinite extension $\mathcal{K}$-trdef of the compact deficiency def by trdef.

Facts (cf. [1]). Let $X$ be a separable metrizable space and $\alpha$ an ordinal number. Then we have the following.
1. If $\alpha = 0$, then $\mathcal{M}(0)-\text{ind} X \leq \mathcal{M}(0)-\text{def} X \leq \mathcal{M}(0)-\text{sur} X$ holds and the converse of the inequalities do not hold. (We notice that $\mathcal{M}(0) = \mathcal{K}$ and so $\mathcal{M}(0)-\text{ind} X = \text{cmpl} X$ and $\mathcal{M}(0)-\text{def} X = \text{def} X$.) We also notice that $\mathcal{A}(0) = \{\emptyset\}$ and hence $\mathcal{A}(0)-\text{ind} X = \mathcal{A}(0)-\text{sur} X$ trivially holds and $\mathcal{A}(0)-\text{def} X$ can not be defined if $X \neq \emptyset$.

2. If $\alpha = 1$, then $\mathcal{A}(1)-\text{ind} X \leq \mathcal{A}(1)-\text{def} X = \mathcal{A}(1)-\text{sur} X$ and $\mathcal{M}(1)-\text{ind} X = \mathcal{M}(1)-\text{def} X \leq \mathcal{M}(1)-\text{sur} X$ hold. The converses of the inequalities above do not hold. (We notice that $\mathcal{A}(1) = \mathcal{S}$ and $\mathcal{M}(1) = \mathcal{C}_0$ and so $\mathcal{M}(1)-\text{ind} X = \text{id} X$.)

3. If $\alpha \geq 2$, then $\mathcal{A}(\alpha)-\text{ind} X = \mathcal{A}(\alpha)-\text{def} X = \mathcal{A}(\alpha)-\text{sur} X$ and $\mathcal{M}(\alpha)-\text{ind} X = \mathcal{M}(\alpha)-\text{def} X = \mathcal{M}(\alpha)-\text{sur} X$ hold.

M. Charalambous [2] showed that the equality $\mathcal{M}(\alpha)-\text{def} X = \mathcal{M}(\alpha)-\text{ind} X$ can not be extended to the transfinite dimension for the case of $\alpha = 1$.

**Example 1.2 (M. Charalambous, [2])** There exists a separable metrizable space $C$ such that $C-\text{trdef} C (= \mathcal{M}(1)-\text{trdef} C) = \omega_0$ and $\text{tricd} C (= \mathcal{M}(1)-\text{trind} C) = \infty$. (We notice that $\mathcal{C}_0-\text{trdef} \leq \text{tricd} X$ holds for every separable metrizable space.)

Thus, it seems to be natural that we ask whether for each ordinal number $\alpha < \omega_1$ there exists a separable metrizable space $X$ such that $\mathcal{M}(\alpha)-\text{trdef} X = \omega_0$ and $\mathcal{M}(\alpha)-\text{trind} X = \infty$ or $\mathcal{A}(\alpha)-\text{trdef} X = \omega_0$ and $\mathcal{A}(\alpha)-\text{trind} X = \infty$.

Connect with the questions above, we have the following.

**Theorem 1.1** Let $\alpha$ be any ordinal with $1 \leq \alpha < \omega_1$.
(1) There exist separable metrizable spaces $X_\alpha, Y_\alpha$ and $Z_\alpha$ such that

(a) $f X_\alpha, f Y_\alpha, f Z_\alpha \leq \omega_0$, where $f$ is either $\text{trdef}$ or $\mathcal{K}$-trisur;

(b) $\mathcal{M}(\alpha)-\text{trind} X_\alpha = -1$ and $\mathcal{A}(\alpha)-\text{trind} X_\alpha = \infty$ (and hence $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)-\text{trind} X_\alpha = \infty$);

(c) $\mathcal{A}(\alpha)-\text{trind} Y_\alpha = -1$ and $\mathcal{M}(\alpha)-\text{trind} Y_\alpha = \infty$ (and hence $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)-\text{trind} Y_\alpha = \infty$);

(d) $\mathcal{M}(\alpha)-\text{trind} Z_\alpha = \mathcal{A}(\alpha)-\text{trind} Z_\alpha = \infty$ and $\mathcal{A}(\alpha + 1) \cap \mathcal{M}(\alpha + 1)-\text{trind} Z_\alpha = -1$. 
(2) There does not exist a separable metrizable space $W_{\alpha}$ such that $A(\alpha)$-trind $W_{\alpha} \neq \infty$, $M(\alpha)$-trind $W_{\alpha} \neq \infty$ and $A(\alpha) \cap M(\alpha)$-trind $W_{\alpha} = \infty$.

**Theorem 1.2** There exists a separable metrizable space $X$ with trdef $X = K$-trsur $X = \omega_{0}$ such that for each $1 \leq \alpha < \omega_{1}$ we have $B$-trind $X = \infty$ and $B$-trdef $X = B$-trsur $X = \omega_{0}$, where $B = A(\alpha), M(\alpha)$ or $A(\alpha) \cap M(\alpha)$.

**Remark 1.1** By Theorems 1.1 and 1.2, it follows that the equalities $M(\alpha)$-def $X = M(\alpha)$-ind $X$ and $A(\alpha)$-sur $X = A(\alpha)$-ind $X$ can not be extended to transfinite-dimensional cases. For the spaces $X_{\alpha}, Y_{\alpha}$ and $Z_{\alpha}$ in Theorem 1.1, we additionally have that

- $M(\alpha)$-trdef $X_{\alpha} = A(\alpha)$-trs $Y_{\alpha} = -1$;
- $M(\alpha)$-trdef $Y_{\alpha} = M(\alpha)$-trdef $Z_{\alpha} = A(\alpha)$-trs $X_{\alpha} = A(\alpha)$-trs $Z_{\alpha} = \omega_{0}$.

We refer the readers to the books [1], [3] and [7] for the dimensions modulo classes, dimension theory and the theory of Borel sets, respectively.

## 2 Outline of proofs.

All classes of topological spaces considered here are additionally assumed to be finitely additive. We will follow some idea of E. Pol [5]. Let $P$ be a class of topological spaces. A space $X$ is said to have the property $(\ast)_{P}$ if for every sequence $\{(A_{i}, B_{i})\}_{i=1}^{\infty}$ of pairs of disjoint compact subsets of $X$ there exist partitions $L_{i}$ between $A_{i}$ and $B_{i}$ in $X$ and an integer $N$ such that $\cap_{i=1}^{N} L_{i} \in P$.

It is evident that the property $(\ast)_{P}$ is closed hereditary.

We have two propositions on the property $(\ast)_{P}$.

**Proposition 2.1** If a space $X$ is covered by a finite family of closed sets such that each element of this cover possesses property $(\ast)_{P}$ then $X$ also possesses this property.

**Proposition 2.2** Let $X$ be a space. If $P$-trind $X \neq \infty$ then $X$ possesses property $(\ast)_{P}$.
Let $\mathbb{I}^\infty = \{(x_i) : 0 \leq x_i \leq 1, i = 1, 2, \ldots\}$ be the Hilbert cube and $Z = \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$ a subspace of the unit interval $\mathbb{I}$. For each $n \geq 1$ we denote the subset $\{(x_i) \in \mathbb{I}^\infty : x_j = 0 \text{ for } j \geq n + 1\}$ of $\mathbb{I}^\infty$ by $\mathbb{I}^n$. For each $n \geq 1$ and each $i = 1, \ldots, n$, we put

$$A_i^n = \{(x_i) \in \mathbb{I}^n \subset \mathbb{I}^\infty : x_i = 0\}, \ B_i^n = \{(x_i) \in \mathbb{I}^n \subset \mathbb{I}^\infty : x_i = 1\}.$$

Choose for each $n \geq 1$ a subset $E_n$ of $\mathbb{I}^n$ and put

$$X = (\{0\} \times \mathbb{I}^\infty) \cup \left( \bigcup_{n=1}^\infty \{\frac{1}{n}\} \times E_n \right). \tag{1}$$

Furthermore, we put $Y = (\{0\} \times \mathbb{I}^\infty) \cup \left( \bigcup_{n=1}^\infty \{\frac{1}{n}\} \times \mathbb{I}^n \right)$. It is clear that $X \subset Y \subset Z \times \mathbb{I}^\infty$, $Y$ is compact, and $Y \setminus X$ is a subspace of the topological sum $\bigoplus_{n=1}^\infty \mathbb{I}^n$. Thus, trind $(Y \setminus X) \leq \omega_0$. Observe also that trind $(X \setminus (\{0\} \times \mathbb{I}^\infty)) \leq \omega_0$. Hence

$$\text{trdef} \ X \leq \omega_0 \text{ and } \mathcal{K}-\text{trsur} \ X \leq \omega_0. \tag{2}$$

**Lemma 2.1** If for each $m \geq 1$ there exists an integer $k(m) \geq m + 1$ such that for any $n \geq k(m)$ and any partition $L_i^n$ between $A_i^n$ and $B_i^n$ in $\mathbb{I}^n$, $i \leq m$, we have $E_n \cap \bigcap_{i=1}^N L_i^n \notin \mathcal{P}$, then $\mathcal{P}$-trind $X = \infty$.

**Proof.** By Proposition 2.2, it suffices to show that $X$ does not have the property $(*)_\mathcal{P}$. For each $i = 1, 2, \ldots$ let $L_i$ be a partition between compact sets $A_i = \{(0, (x_j)) \in \{0\} \times \mathbb{I}^\infty : x_i = 0\}$ and $B_i = \{(0, (x_j)) \in \{0\} \times \mathbb{I}^\infty : x_i = 1\}$ We shall show that $\bigcap_{i=1}^N L_i \notin \mathcal{P}$ for every natural number $N$. Let $N$ be a natural number. For each $i \geq 1$ let us consider a partition $L_i'$ between $A_i$ and $B_i$ in $Y$ such that $L_i = L_i' \cap X$. Note that for every $i$ there exists a natural number $n_i \geq 2$ such that for any $n \geq n_i$ $L_i^n = L_i' \cap (\{\frac{1}{n_i}\} \times \mathbb{I}^n)$ is a partition between $\{\frac{1}{n_i}\} \times A_i^n$ and $\{\frac{1}{n_i}\} \times B_i^n$ in $\{\frac{1}{n_i}\} \times \mathbb{I}^n$. Let $n$ a fixed integer with $n \geq \max\{n_1, \ldots, n_N, k(N)\}$. Then $C = (\bigcap_{i=1}^N L_i^n) \cap (\{\frac{1}{n}\} \times E_n) = (\bigcap_{i=1}^N L_i) \cap (\{\frac{1}{n}\} \times E_n)$ is a closed subset of $\bigcap_{i=1}^N L_i$, and $C \notin \mathcal{P}$ by the assumption. So $\bigcap_{i=1}^N L_i \notin \mathcal{P}$.

We shall also use the following.

**Lemma 2.2** ([8, Lemma 5.2]) Let $L_{ij}$ be partitions between the opposite faces $A_{ij}^n$ and $B_{ij}^n$ in $\mathbb{I}^n$, where $1 \leq i_1 < i_2 \ldots < i_p \leq n$ and $1 \leq p < n$. Then for any $k \neq i, j = 1, \ldots, p$, there is a continuum $C \subset \bigcap_{j=1}^p L_{ij}$ meeting the faces $A_k^n$ and $B_k^n$. 

Lemma 2.3 Let $\alpha$ be an ordinal number with $1 \leq \alpha < \omega_1$. Then there exist subsets $Q_\alpha$, $P_\alpha$ and $D_\alpha$ of $\mathbb{I}$ such that

1. $Q_\alpha \in A(\alpha) - M(\alpha)$,
2. $P_\alpha \in M(\alpha) - A(\alpha)$,
3. $D_\alpha \in A(\alpha+1) \cap M(\alpha+1) - (A(\alpha) \cup M(\alpha))$.

Proof of Theorem 1.1. (1) We shall prove for $Y_\alpha$ only. We put

$$Y_\alpha = \left(\{0\} \times \mathbb{I}^\infty\right) \cup \left(\bigcup_{n=2}^{\infty} \left\{\frac{1}{n}\right\} \times \pi_n^{-1}(Q_\alpha)\right),$$

where $Q_\alpha$ is the subspace $\mathbb{I}$ described in Lemma 2.3 and $\pi_n : \mathbb{I}^n \to \mathbb{I}$ be the projection onto the $n$-th factor. By the construction of $Y_\alpha$, it is clear that $M(\alpha)$-trdf $Y_\alpha \leq \operatorname{trdf} Y_\alpha \leq \omega_0$, and $M(\alpha)$-trsur $Y_\alpha \leq \omega_0$. Since the absolute Borel classes are preserved under perfect preimages, it follows that $\pi_n^{-1}(Q_\alpha) \in A(\alpha)$. Thus, $Y_\alpha \in A(\alpha)$ and hence $A(\alpha)$-trind $Y_\alpha = -1$. Now, it suffices to show that $M(\alpha)$-trind $Y_\alpha = \infty$. To apply Lemma 2.1, for every natural number $m$ let $k(m) = m + 1$. For each $n \geq k(m)$ and each $i \leq n$ let $L_i^n$ be a partition between $A_i^n$ and $B_i^n$ in $\mathbb{I}^n$. By Lemma 2.2, there exists a continuum $C$ such that $C \subset \bigcap_{i=1}^{n} L_i^n$ and $C \cap A_i^n \neq \emptyset \neq C \cap B_i^n$. Let $\pi_n^C = \pi|C : C \to \mathbb{I}$ be the restriction of the projection $\pi_n$ over $C$. Then $C \cap \pi_n^{-1}(Q_\alpha) = (\pi_n^C)^{-1}(Q_\alpha) \subset \bigcap_{i=1}^{n} L_i^n \cap \pi_n^{-1}(Q_\alpha)$. Since $C \cap \pi_n^{-1}(Q_\alpha)$ is closed set of $\bigcap_{i=1}^{n} L_i^n \cap \pi_n^{-1}(Q_\alpha)$ and $(\pi_n^C)^{-1}(Q_\alpha) \notin M(\alpha)$, it follows that $\bigcap_{i=1}^{n} L_i^n \cap \pi_n^{-1}(Q_\alpha) \notin M(\alpha)$. Thus, it follows from Lemma 2.1 that $M(\alpha)$-trind $Y_\alpha = \infty$. This completes the proof.

(2) The second part of Theorem 1.1 is a direct consequence of the following proposition.

Proposition 2.3 Let $X$ be a separable metrizable space with $A(\alpha)$-trind $X \leq \mu_1$ and $M(\alpha)$-trind $X \leq \mu_2$. Then

$$A(\alpha) \cap M(\alpha)$$

$$\begin{cases} \mu_1 + n(\mu_2) + 1, & \text{if } \lambda(\mu_1) = \lambda(\mu_2), \\ \mu_1, & \text{if } \lambda(\mu_1) > \lambda(\mu_2). \end{cases}$$

Proof. The proposition can be proved by a standard transfinite induction on $\nu = \max\{\mu_1, \mu_2\}$.

Connect with Proposition 2.1, we ask the following question.
**Question 2.1** Does there exist a separable metrizable space $X_\alpha$ such that $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha) \text{-trind } X_\alpha > \max\{\mathcal{A}(\alpha) \text{-trind } X_\alpha, \mathcal{M}(\alpha) \text{-trind } X_\alpha\}$ for each ordinal number $\alpha$? In particular, does there exist a separable metrizable space $X$ such that $C_0 \cap \mathcal{S} \text{-ind } X = 1$ and $C_0 \text{-ind } X = \mathcal{S} \text{-trind } X = 0$?

Recall from M.G. Charalambous ([2]) that we call a subset $A$ of a space $X$ a Bernstein set if $|A \cap B| = |(X \setminus A) \cap B| = c$ for every uncountable Borel set $B$ of $X$, where $c$ denotes the cardinality of the continuum. It is known that every uncountable completely metrizable space $X$ has countably many disjoint Bernstein sets. We notice that $A \notin AB$ for every Bernstein set $A$ of an uncountable completely metrizable space $X$.

**Proof of Theorem 1.2.** Let $F$ be a Bernstein set of $\mathbb{I}$. We put $X = (\{0\} \times \mathbb{I}^\infty) \cup (\bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times \pi_n^{-1}(F))$. Then, we can show that $X$ is the desired space by an argument similar to Theorem 1.1.

Connect with Theorem 1.1, we may ask the following question.

**Question 2.2** For each ordinal numbers $\alpha$ and $\beta$ with $1 \leq \alpha < \omega_1$ and $0 \leq \beta < \omega_1$ do there exist separable metrizable spaces $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ which satisfy the following conditions?

1. $\mathcal{A}(\alpha) \text{-trind } X_{\alpha,\beta} = \beta$,
2. $\mathcal{M}(\alpha) \text{-trind } Y_{\alpha,\beta} = \beta$, and
3. $\mathcal{M}(\alpha) \text{-trind } X_{\alpha,\beta} = \mathcal{A}(\alpha) \text{-trind } Y_{\alpha,\beta} = -1$.

**References**


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