HAUSDORFF HYPERSPACES OF EUCLIDEAN SPACES
AND THEIR DENSE SUBSPACES

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Here, we introduce the results obtained in the paper [11] and related problems. We consider metric spaces and their hyperspaces endowed with the Hausdorff metric. Specifically, given a metric space $X = (X, d)$, we shall denote by $\text{Cld}(X)$ and $\text{Bd}(X)$ the hyperspaces consisting of all nonempty closed sets and of all nonempty bounded closed sets in $X$ respectively and we denote by $d_H$ the Hausdorff metric, which is infinite-valued on $\text{Cld}(X)$ if $X$ is unbounded. When $X$ is compact, the space $\text{Cld}(X) (= \text{Bd}(X))$ is equal to the hyperspace $\exp(X)$ of all nonempty compact sets with the Vietoris topology. Even if $X$ is noncompact, on the space $\exp(X)$, the Hausdorff metric topology coincides with the Vietoris topology. However, in case $X$ is noncompact, these topologies are very different on the spaces $\text{Cld}(X)$ and $\text{Bd}(X)$.

Vietoris hyperspaces $\exp(X)$ have been studied extensively for many years. Among the known results, let us mention the theorem of Curtis and Schori [8] (cf. [13, Chapter 8]), saying that $\exp(X)$ is homeomorphic to $(\cong)$ the Hilbert cube $Q = [-1, 1]^{\omega}$ if and only if $X$ is a Peano continuum, that is, it is compact, connected and locally connected. Later, Curtis [7] characterized non-compact metric spaces $X$ for which $\exp(X)$ is homeomorphic to the Hilbert cube minus a point $Q \setminus 0 (= Q \setminus \{0\})$ or the pseudo-interior $s = (-1, 1)^{\omega}$ of $Q$. In particular, $\text{Bd}(\mathbb{R}^m) = \exp(\mathbb{R}^m)$ is homeomorphic to $Q \setminus 0$. For more information concerning Vietoris hyperspaces, we refer to the book of Ilanes and Nadler [10].

It is well known that the hyperspace $\exp(X)$ is an ANR (AR) if and only if $X$ is locally connected (and connected). On the other hand, it is proved in [6] that the space $\text{Bd}(X)$ is an ANR (AR) whenever the metric on $X$ is almost convex, that is,

\footnote{It is well known that $s$ is homeomorphic to the separable Hilbert space $\ell_2$.}
for every $\alpha > 0, \beta > 0$ and for every $x, y \in X$ such that $d(x, y) < \alpha + \beta$, there exists $z \in X$ with $d(x, z) < \alpha$ and $d(z, y) < \beta$. This condition was further weakened in [12], which has turned out to be actually a necessary and sufficient one by Banakh and Voytsitskyy [3]. In the last paper, several equivalent conditions are given, which are too technical to mention them here. We refer to [3] for the details. On the other hand, Cld($X$) is not connected whenever $X$ is a metric space which is not totally bounded. For example, Cld($\mathbb{R}$) has $2^{\aleph_0}$ many components.

The completion of a metric space $X = \langle X, d \rangle$ is denoted by $\tilde{X} = \langle \tilde{X}, d \rangle$. Then $\text{Bd}(X)$ can be identified with the subspace of $\text{Bd}(\tilde{X})$, via the isometric embedding $A \mapsto \text{cl}_{\tilde{X}} A$. Thus we shall often write $\text{Bd}(X) \subseteq \text{Bd}(\tilde{X})$, having in mind this identification. In this case, $\text{Bd}(\tilde{X})$ is the completion of $\text{Bd}(X)$. By such a reason, we also consider a dense subspace $D$ of a metric space $X = \langle X, d \rangle$. For each $0 \leq k < m$, let

$$\nu_k^m = \{x = (x_i)_{i=1}^m \in \mathbb{R}^m : x_i \in \mathbb{R} \setminus \mathbb{Q} \text{ except for at most } k \text{ many } i\},$$

which is the universal space for completely metrizable subspaces in $\mathbb{R}^m$ of dim $\leq k$. In case $2k + 1 < m$, $\nu_k^m$ is homeomorphic to the $k$-dimensional Nöbeling space $\nu_k^{2k+1}$, which is the universal space for all separable completely metrizable spaces. Note that $\nu_0^m = (\mathbb{R} \setminus \mathbb{Q})^m \cong \mathbb{R} \setminus \mathbb{Q}$.

**Theorem 1.** Suppose $\langle m, k \rangle = \langle 1, 0 \rangle$ or $0 \leq k < m - 1$. Then,

$$\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(\nu_k^m) \rangle \cong \langle Q \setminus 0, s \setminus 0 \rangle.$$

Consequently, $\text{Bd}(\nu_k^m) \cong \ell_2$.

This can be derived from the following:

**Theorem 2.** Let $D$ be a dense $G_\delta$ set in $\mathbb{R}^m$ such that $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$ and in case $m > 1$ it is assumed that $D = p[D] \times \mathbb{R}$, where $p : \mathbb{R}^m \to \mathbb{R}^{m-1}$ is the projection onto the first $m - 1$ coordinates. Then, $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(D) \rangle \cong \langle Q \setminus 0, s \setminus 0 \rangle$.

**Question 1.** In case $m > 1$, under the only assumption that $D \subseteq \mathbb{R}^m$ is a dense $G_\delta$ set and $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$, is the pair $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(D) \rangle$ homeomorphic to $\langle Q \setminus 0, s \setminus 0 \rangle$? In particular, is the pair $\langle \text{Bd}(\mathbb{R}^m), \text{Bd}(\nu_m^m) \rangle$ homeomorphic to $\langle Q \setminus 0, s \setminus 0 \rangle$?

We also consider the following dense subspaces of $\text{Bd}(X)$:

- $\text{Nwd}(X)$ — all nowhere dense closed sets;
- $\text{Perf}(X)$ — all perfect sets;\(^2\)

\(^2\)I.e., completely metrizable closed sets which are dense in itself.
• Cantor(\(X\)) — all compact sets homeomorphic to the Cantor set.

In case \(X = \mathbb{R}^m\), we can also consider the following subspace:

• \(\mathcal{N}(\mathbb{R}^m)\) — all closed sets of the Lebesgue measure zero.

For these spaces, we have the following:

**Theorem 3.** Let \(\mathcal{F}\) be one of the following subspaces of \(\text{Bd}(\mathbb{R}^m)\):

\[
\text{Nwd}(\mathbb{R}^m), \text{Perf}(\mathbb{R}^m), \text{Cantor}(\mathbb{R}^m) \text{ and } \mathcal{N}(\mathbb{R}^m).
\]

Then, \(\langle \text{Bd}(\mathbb{R}^m), \mathcal{F} \rangle \cong \langle Q \setminus 0, s \setminus 0 \rangle\), hence \(\mathcal{F} \cong \ell_2\).

To prove Theorems 2 and 3 above, we adopt the characterization of the pseudo-boundary \(Q \setminus s\) of the Hilbert cube \(Q\), see [5].

We also study the space \(\text{Cld}(\mathbb{R})\). It is very different from the hyperspace \(\exp(\mathbb{R})\). It is not hard to see that \(\text{Cld}(\mathbb{R})\) has \(2^{\aleph_0}\) many components, \(\text{Bd}(\mathbb{R})\) is the only separable one and any other component has weight \(2^{\aleph_0}\). Applying Toruńczyk's Characterization of Hilbert space [14] (cf. [15]), we can prove

**Theorem 4.** Let \(\mathcal{H}\) be a nonseparable component of \(\text{Cld}(\mathbb{R})\) which does not contain \(\mathbb{R}, [0, +\infty), (-\infty, 0]\). Then \(\mathcal{H} \cong \ell_2(2^{\aleph_0})\).

**Question 2.** Does Theorem 4 hold even if \(\mathcal{H}\) contains \(\mathbb{R}, [0, \infty)\) or \((-\infty, 0]\)?

**Question 3.** For \(m > 1\), is \(\text{Cld}(\mathbb{R}^m) \setminus \text{Bd}(\mathbb{R}^m)\) an \(\ell_2(2^{\aleph_0})\)-manifold?

Now, we consider the subspaces \(\mathcal{N}(\mathbb{R}), \text{Nwd}(\mathbb{R}), \text{Perf}(\mathbb{R})\) and \(\text{Cld}(\mathbb{R} \setminus Q)\) of \(\text{Cld}(\mathbb{R})\). Similarly to \(\text{Bd}(\mathbb{R})\), it can be shown that those complements are \(Z_\sigma\)-sets in \(\text{Cld}(\mathbb{R})\).

Due to Negligibility Theorem ([1], [9]), if \(M\) is an \(\ell_2(2^{\aleph_0})\)-manifold and \(A\) is a \(Z_\sigma\)-set in \(M\) then \(M \setminus A \cong M\). Thus, the following follows from Theorem 4:

**Corollary 5.** Let \(\mathcal{H}\) be a nonseparable component of \(\text{Cld}(\mathbb{R})\) which does not contain \(\mathbb{R}, [0, +\infty), (-\infty, 0]\). Then, the following spaces are homeomorphic to \(\ell_2(2^{\aleph_0})\):

\[
\mathcal{H} \cap \mathcal{N}(\mathbb{R}), \mathcal{H} \cap \text{Nwd}(\mathbb{R}), \mathcal{H} \cap \text{Perf}(\mathbb{R}) \text{ and } \mathcal{H} \cap \text{Cld}(\mathbb{R} \setminus Q).
\]

**Borel classes.** Given a metric space \(\langle X, d \rangle\), let \(\langle \tilde{X}, d \rangle\) be its completion. Then, the hyperspace \(\text{Bd}(\tilde{X})\) is the completion of the hyperspace \(\text{Bd}(X)\). Concerning Borel classes of hyperspaces, the following are also shown in the paper [11]:

1. \(\text{Bd}(X)\) is \(F_{\sigma\delta}\) in \(\text{Bd}(\tilde{X})\) if \(X\) is \(\sigma\)-compact.
2. \(\text{Bd}(X)\) is \(G_\delta\) in \(\text{Bd}(\tilde{X})\) if \(X\) is Polish.\(^3\)

\(^3\)I.e., separable and completely metrizable
(3) $\text{Bd}(X)$ is Polish for every Polish space $X$ in which bounded sets are totally bounded.

(4) $\text{Nwd}(X)$ is $G_\delta$ in $\text{Bd}(X)$ for every separable metric space $X$.

(5) $\text{Perf}(X)$ is $G_\delta$ in $\text{Bd}(X)$ if $X$ is separable and locally compact.

(6) $\text{Perf}(X)$ is $F_{\sigma\delta}$ in $\text{Bd}(X)$ for every Polish space $X$.

(7) $\text{Bd}(X)$ is analytic for every analytic metric space $X$ in which bounded sets are totally bounded.

Fix a dense set $X$ in a separable Banach space $E$ which admits the metric $d$ induced from the norm of $E$. Then $(X,d)$ is an almost convex metric space and therefore by a result of [6] the space $\text{Bd}(X)$ is an AR. In case $X$ is $G_\delta$, the space $\text{Bd}(X)$ is completely metrizable by (2). If additionally $E$ is finite-dimensional then $\text{Bd}(X)$ is Polish by (3). In case $X$ is $\sigma$-compact, by (1), $\text{Bd}(X)$ is absolutely $F_{\sigma\delta}$.

**Remarks.** Recently, Banakh and Voytsitskyy [4] proved that the space $\text{Cld}(X)$ (resp. $\text{Bd}(X)$) is homeomorphic to $\ell_2$ if and only if $X$ is a completely metrizable nowhere locally compact metric space such that each (resp. bounded) subset of $X$ is totally bounded and the completion $\breve{X}$ of $X$ is connected and locally connected.

**References**


