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Teichmüller groupoids and number theory

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1. Introduction

In the title, Teichmüller groupoids are the fundamental groupoids of moduli spaces of (algebraic) curves, and number theory means studying the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ over $\mathbf{Q}$. These subjects are combined by Grothendieck's "Esquisse d'un Programme" [G]. The aim of this note is to show his assertion:

Let g, n be non-negative integers such that $2g-2+n>0$, and let $M_{g,n}$ be the moduli stack over $\overline{\mathbf{Q}}$ of n-pointed proper smooth curves of genus g. Then its algebraic fundamental groupoid $\widehat{\pi}_1(M_{g,n}; a, b)$ for rational points a, b has natural $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-action, and this is generated by the algebraic fundamental groupoids of the basic objects $M_{0,4}, M_{1,1}, M_{0,5}, M_{1,2}$ of dimension ≤ 2 together with the Galois action.

Denote by $\pi_1(M_{g,n}(\mathbf{C}); a, b)$ the topological fundamental groupoid representing homotopy classes of paths in $M_{g,n}(\mathbf{C})$ from a to b which becomes a torsor over the Teichmüller modular group (or mapping class group). Then by a result of Oda [O], the profinite completion of $\pi_1(M_{g,n}(\mathbf{C}); a, b)$ becomes $\widehat{\pi}_1(M_{g,n}; a, b)$, and hence one can describe it as the set of etale paths from a to b, and consider the Galois action.

First, we recall Grothendieck's "proof" whose completed version cannot be found by the author regrettably. Let $\overline{M}_{g,n}$ be the Deligne-Mumford-Knudsen compactification classifying n-pointed stable curves of genus g. Then the complement $D_{g,n} = \overline{M}_{g,n} - M_{g,n}$ consisting of singular curves is the union of the images by the natural map from $\overline{M}_{g-1,n+2}$ and from $\overline{M}_{g_1,n_1} \times \overline{M}_{g_2,n_2}$, where $g_1 + g_2 = g$, $n_1 + n_2 = n+2$. Therefore, if

$\dim(M_{g,n}) = 3g - 3 + n > 2$, then a "theorem of Lefschetz type" should hold because $M_{g,n}$ is not too near complete, and hence

$$\widehat{\pi}_1\left(M_{g,n}\right) \cong \widehat{\pi}_1\left(\text{a tubular neighborhood of } D_{g,n}\right)$$

which is, by Van Kampen's theorem, generated by Dehn twists associated with degeneration processes and by $\widehat{\pi}_1\left(M_{g',n'}\right)$, where $g' \leq g$ and $\dim(M_{g',n'}) < \dim(M_{g,n})$.

2. Game of Lego-Teichmüller

Our approach to prove Grothendieck's assertion is topology and arithmetic on a "game of Lego-Teichmüller" which is also indicated in the Esquisse. First, we review a topological game of Lego-Teichmüller. It has a long history (may be) starting from Hatcher-Thurston's paper [HaT], and was developed by many mathematicians and physists including Moore and Seiberg [MS], Funar and Gelca [FuG], Bakalov and Kirillov [BK1, 2], Hatcher, Lochack and Schneps [HaLS] and Hiroaki Nakamura [N] (see also [F]). Here we follow Nakamura's formulation and result. We call a 3-holed sphere with 3 seams a "quilt", and consider quilt decompositions of a pointed Riemann surface as a refinement of pants decompositions to fit seams to each other. Then the extended Hatcher complex of type (g, n) is defined as the cell complex whose

- 0-cells are isotopy classes of quilt decompositions of a fixed n-pointed Riemann surface of genus g,
- 1-cells are the following elementary moves of 3-types:
  - fusing (or Associative, A-)moves connecting different sewing processes from two 3-holed spheres to one 4-holed sphere,
  - simple (or S-)moves connecting different sewing processes from one 3-holed spheres to one 1-holed real surface of genus 1,
  - Dehn half-twists,
- 2-cells are relations induced from the bacis objects $M_{0,4}$, $M_{1,1}$, $M_{0,5}$ and $M_{1,2}$ (for example, the pentagon relation is induced from $M_{0,5}$).

Theorem 1 (Nakamura [N]). *The extended Hatcher complex of type (g, n) is connected and simply connected.*

Since the Teichmüller modular group acts on the extended Hatcher complex faithfully, one can see that any topological Teichmüller groupoid is generated by the fundamental groupoids of the basic objects.

Second, we review an arithmetic game of Lego-Teichmüller. Here we consider a quilt as a 3-holed $\mathbf{P}^1_{\mathbf{C}}$ around $0, 1, \infty$ with 3 real lines. Then by gluing holes in several quilts to fit seams to each other (like the Lego game!), we have a real deformation of a maximally degenerate pointed curve. Furthermore, using arithmetic Schottky uniformization theory given in [IhN, I1], we can show that this deformation can be constructed over the ring consisting of polynomials of moduli parameters and of power series of deformation parameters over $\mathbf{Z}$, and that the elementary moves are described by moving these parameters. Therefore, we have:

Theorem 2 (cf. [I2]). *There exists an appropriate base set $\mathcal{L} \subset M_{g,n}(\mathbf{C})$ of the Teichmüller groupoid of $M_{g,n}$ consisting of fusing moves and simple moves. For the natural $\mathbf{Z}$-structure of $M_{g,n}$, $\mathcal{L}$ is a real orbifold of dimension $3g-3+n$ in the real locus, and gives $\mathbf{Z}$-rational tangential base points ($\doteq$ unit tangent vectors) around the points at infinity corresponding to maximally degenerate n-pointed curves of genus g. If $(g, n) = (0, 4)$, then*

$$\mathcal{L} = \mathbf{R} - \{0, 1\} \subset M_{0,4}(\mathbf{C}) = \mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\},$$

and if $(g, n) = (1, 1)$, then

$$\mathcal{L} = \text{Image of } \sqrt{-1}\, \mathbf{R}_{>0} \subset M_{1,1}(\mathbf{C}) = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}/SL_2(\mathbf{Z}).$$

For general (g, n), $\mathcal{L} \subset M_{g,n}(\mathbf{C})$ is constructed by gluing $\mathcal{L}$ in $M_{0,4}(\mathbf{C})$, $M_{1,1}(\mathbf{C})$ using the arithmetic Schottky uniformization theory.

By these theorems and a result of Anderson-Ihara [AI], we can show Grothendieck's assertion by calculating the Galois action on the elementary moves in terms of that on $\widehat{\pi}_1$ of the basic objects as follows:

Theorem 3 (cf. [I2]). *The action of* $\mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ *on generators of the profinite Teichmüller groupoid of $M_{g,n}$ can be described as*

- *the action on fusing moves = the action on* $\mathcal{L} \subset M_{0,4}(\mathbf{C})$,
- *the action on simple moves = the action on* $\mathcal{L} \subset M_{1,1}(\mathbf{C})$,
- *the action on Dehn half-twists is given by the cyclotomic character.*

3. Applications

First, we mention a result of Lochack, Nakamura and Schneps [LNS, NS] on a refinement $\mathbb{I}\Gamma$ of the profinite Grothendieck-Teichmüller group

$$\begin{aligned} \widehat{GT} &= \text{Aut}\,(\text{profinite Teichmüller groupoids of degree } 0) \\ &\subset \text{Aut}\,(\widehat{\pi}_1(M_{0,4})) \end{aligned}$$

introduced by Drinfeld [Dr]. By adding relations induced from $M_{1,1}, M_{1,2}$, they defined a subgroup $\mathbb{I}\Gamma$ of $\widehat{GT}$, and showed that $\mathbb{I}\Gamma$ acts on the profinite Teichmüller modular groups extending the natural Galois action. By Theorems 1–3, $\mathbb{I}\Gamma$ becomes the automorphism group of the "profinite Teichmüller tower" which is the whole system of all profinite Teichmüller groupoids, and that the action of $\mathbb{I}\Gamma$ on the profinite Teichmüller tower is an extension of the Galois action. From this fact and a result of Belyi [Be], we get the following simple picture:

$$\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \subset \mathbb{I}\Gamma = \text{Aut}\,(\text{the profinite Teichmüller tower})\,.$$

The author does not know whether $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \mathbb{I}\Gamma$ holds or not.

Second, we review an application of our result to conformal field theory. Using Theorems 1 and 2, we can give a mathematical translation of Moore-Seiberg's work [MS]. Furthermore, we can calculate the monodromy representation of the Teichmüller groupoid for Tsuchiya-Ueno-Yamada's conformal field theory [TUY] as follows:

Theorem 4 (cf. [I3]). *The monodromy representation of the Teichmüller groupoid of* $M_{g,n}$ *associated with the TUY-theory can be described as follows:*

- *monodromy of fusing moves*
  = connection matrices of the Knizhnik-Zamolodchikov equation [Dr],

- *monodromy of simple moves*
  $=$ *transformation matrices of non-abelian theta functions [KP],*

- *monodromy of Dehn half-twists*
  $= \exp\left(\pi\sqrt{-1} \times (\textit{residues of the connection})\right).$

Consequently, the monodromy for TUY is given as the monodromy for the Wess-Zumino-Witten model described by Kohno [Ko].

4. Concluding remarks

The author thinks that main targets in number theory and in physics are $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and the UNIVERSE respectively, which seem to be connected as:

$$\begin{tikzcd} \boxed{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})} \arrow[dr, dash, "\text{Grothendieck's program}"'] & & \boxed{\text{UNIVERSE}} \arrow[dl, dash, "\text{string theory}"] \\ & \boxed{\text{Moduli of curves}} & \end{tikzcd}$$

Roughly speaking, these correspond to:

$$\begin{tikzcd} \boxed{\text{Etale aspect}} \arrow[dr, dash] & & \boxed{\text{De Rham aspect}} \arrow[dl, dash] \\ & \boxed{\pi_1 \text{ of the Moduli}} & \end{tikzcd}$$

The author does not know about a motivic theory on π_1 of any moduli of curves, however our results seem to suggest that a motivic theory on $\pi_1(M_{g,n})$ (if it exists!) can be reduced to that on π_1 of the basic objects. Deligne [D], Deligne-Goncharov [DG] and others constructed a motivic theory on the nilpotent quotients of $\pi_1(M_{0,4})$ as mixed Tate motives. This etale realization gives rise to Soule's characters (cf. [Ih], [HM]), and de Rham realization gives rise to multiple zeta values.

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