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CYCLIC VECTORS IN FOCK-TYPE SPACES

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1. Introduction

Let \( \mathbb{D} \) be the open unit disk of the complex plane \( \mathbb{C} \). We denote the polynomial ring by \( C \), and the space of all entire functions by \( Hol(\mathbb{C}) \). Let \( X \) be a complete semi-normed space of holomorphic functions on a domain \( \Omega \) in \( \mathbb{C} \). For a subset \( E \) of \( X \), let \( \overline{E} \) be the closure of \( E \) in \( X \). A function \( f \) is said to be cyclic in \( X \) if \( fC \subset X \) and \( \overline{fC} = X \).

In the Hardy spaces \( H^p(\mathbb{D}) (0 < p < \infty) \), it is well known that a function is cyclic if and only if it is \( H^p(\mathbb{D}) \)-outer (see [Gar]). Also in the Bergman spaces \( L^p_a(\mathbb{D}) (0 < p < \infty) \), it is known that a function is cyclic if and only if it is \( L^p_a(\mathbb{D}) \)-outer (see [HKZ]). Recently the author has characterized the cyclic vectors in the classical Fock space. The classical Fock space \( L^2_a(\mathbb{C}) \) is

\[
L^2_a(\mathbb{C}) = \left\{ f \in Hol(\mathbb{C}) : \| f \|_{L^2_a(\mathbb{C})} = \left\{ \int_{\mathbb{C}} |f(z)|^2 d\mu(z) \right\}^{1/2} < \infty \right\}
\]

where

\[
d\mu(z) = e^{-|z|^2} \frac{dA(z)}{2\pi}
\]

is the Gaussian measure on \( \mathbb{C} \) and \( dA \) is the ordinary Lebesgue measure. In [Izu1], we have proved the following:

**Theorem A.** Let \( h(z) \in Hol(\mathbb{C}) \). Then the following are equivalent:

(i) \( f(z) \) is a nonvanishing function in \( L^2_a(\mathbb{C}) \).

(ii) \( f(z) = e^{h(z)} \), \( h(z) = \alpha z^2 + \beta z + \gamma \) for \( \alpha, \beta, \gamma \in \mathbb{C} \), \( |\alpha| < \frac{1}{4} \).

(iii) \( f(z) \) is cyclic in \( L^2_a(\mathbb{C}) \).

It is known that there are non-vanishing functions in \( H^p(\mathbb{D}) \) and \( L^p_a(\mathbb{D}) \) which are not cyclic in the respective spaces (see [Gar] and [HKZ]). In fact, Brown and Shields posed the following question [BS]:

**Question B.** Let \( \Omega \) be bounded region in \( \mathbb{C} \). Does there exist a polynomially dense Banach space \( X \) of analytic functions in \( \Omega \) with the two properties?
(i) $zX \subseteq X$
(ii) for any $\lambda \in \Omega$, point evaluation functional for $\lambda$ is bounded, in which a function $f(z)$ is cyclic if and only if $f(z) \neq 0$ for all $z \in \Omega$?

The above theorem is not the answer of this question. But it says that there exists a polynomially dense Banach space in which every non-vanishing function is cyclic.

In this paper, we consider the cyclic vectors in more generalized spaces.

Let $0 < p < \infty$, $s > 0$ and $\alpha > 0$. Let $\phi$ be a positive function on $[0, \infty)$. The space $L^p_{\alpha}(\mathbb{C}, \phi)$ consists of those entire functions whose semi-norm

$$
\|f\|_{L^p_{\alpha}(\mathbb{C}, \phi)} = \left\{ \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} dA(z) \right\}^{1/p}
$$

is finite. This space is called Fock-type space. Throughout this paper, we put $\phi(|z|) = \frac{\alpha}{p} |z|^s$. We study the cyclic vectors in $L^p_{\alpha}(\mathbb{C}, \phi)$.

This is a summary of the paper [Izu2].

2. Results

The following is our main result:

**Theorem 1.** Let $f$ be a function in $L^p_{\alpha}(\mathbb{C}, \phi)$ satisfying $fC \subseteq L^p_{\alpha}(\mathbb{C}, \phi)$. Then the following are equivalent:

(i) $f(z)$ is a non-vanishing function.
(ii) $f(z) = e^{h(z)}$ for $h(z) = \sum_{k=0}^{[s]} a_k z^k$, $a_k \in \mathbb{C}$, where $[s]$ is the largest integer with $[s] \leq s$, and in addition $|a_s| < \frac{\alpha}{p}$ if $s$ is an integer.
(iii) $f(z)$ is cyclic in $L^p_{\alpha}(\mathbb{C}, \phi)$.

We know that every non-vanishing function in the classical Fock space $L^2_{\alpha}(\mathbb{C})$ is cyclic. In our case, we notice that it is not valid for some positive numbers; that is, if $s$ is not an integer or $s = 1, 2, 3, 4$, then $L^p_{\alpha}(\mathbb{C}, \phi)$ has the same property as the one in $L^2_{\alpha}(\mathbb{C})$, but if $s = 5, 6, 7, \cdots$, the situation is different. For example, although $f(z) = e^{\beta z^s}$ is a non-vanishing function in $L^p_{\alpha}(\mathbb{C}, \phi)$, the function $f(z)$ does not satisfy $fC \subseteq L^p_{\alpha}(\mathbb{C}, \phi)$. Obviously this function $f(z)$ is not cyclic. But if we consider the non-vanishing functions just satisfying $fC \subseteq L^p_{\alpha}(\mathbb{C}, \phi)$, then the situation is similar.
To prove Theorem 1, we introduce the space $\mathcal{F}_\phi^p$ which is studied in [MMO]. The space is

$$
\mathcal{F}_\phi^p = \left\{ f \in Hol(\mathbb{C}) : \|f\|_{\mathcal{F}_\phi^p}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} \rho^{-1} \phi dA(z) < \infty \right\}
$$

where $\Delta \phi$ is the Laplacian of $\phi$ and $\rho^{-1} \Delta \phi$ is a regular version of $\Delta \phi$. If $p = 2$, then $\mathcal{F}_\phi^2$ is a Hilbert space with inner product

$$
\langle f, g \rangle_{\mathcal{F}_\phi^2} = \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-2\phi(z)} \rho^{-1} \Delta \phi dA(z).
$$

We denote the reproducing kernel of $\mathcal{F}_\phi^2$ by $K_\lambda$, $\lambda \in \mathbb{C}$. The following lemma is proved by Marco, Massaneda and Ortega-Cerdà in [MMO, Lemma 21].

**Lemma 2.** There exists a positive number $C$ such that for any $\lambda \in \mathbb{C}$

$$
C^{-1} e^{2\phi(\lambda)} \leq \|K_\lambda\|_{\mathcal{F}_\phi^2}^2 \leq C e^{2\phi(\lambda)}.
$$

In [CGH], Chen, Guo and Hou proved the following:

**Lemma 3.**

$$
\lim_{|\lambda| \to \infty} \frac{\langle f, K_\lambda \rangle_{\mathcal{F}_\phi^2}}{\|K_\lambda\|_{\mathcal{F}_\phi^2}^2} = 0
$$

for any $f \in \mathcal{F}_\phi^2$.

By Lemma 2 and 3, we get the following:

**Lemma 4.** The following are equivalent:

(i) $f(z) \in L_a^p(\mathbb{C}, \phi)$ is a non-vanishing function satisfying $f \subset L_a^p(\mathbb{C}, \phi)$.

(ii) $f(z) = e^{h(z)}$ where $h(z) = \sum_{k=0}^{[s]} a_k z^k$ and in addition $|a_s| < \frac{\alpha}{p}$ if $s$ is an integer.

By Lemma 4, (i)$\Leftrightarrow$(ii) in Theorem 1 has been proved.

The following two lemmas are the generalizations of the results in [GW].

**Lemma 5.** Let $f \in L_a^p(\mathbb{C}, \phi)$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then we have the following:

(i) There exists a constant $C_1 > 0$, which depends on $f$, satisfying

$$
|c_n| \leq C_1 e^{2-s} \left( \frac{s \alpha e}{pn+2-s} \right)^{\frac{n}{s}} \|f\|_{L_a^p(\mathbb{C}, \phi)}.
$$
(ii) For large $n$,
\[
\|z^n\|_{L_a^p(\mathbb{C}, \phi)}^p = \frac{\alpha^{-\frac{pn+2}{s} \frac{\tau \iota + 2}{s}}}{s} \Gamma\left(\frac{pn+2}{s}\right)
\sim \frac{1}{s \alpha} \left(2\pi \frac{pn+2-s}{s}\right)^{\frac{1}{2}} \left(\frac{pn+2-s}{s \alpha e}\right)^{\epsilon \frac{7 \iota + 2-s}{s}},
\]
where $\Gamma$ denotes the gamma function.

(iii) There is a constant $C_2 > 0$, which depends on $f$, satisfying
\[
\|c_n z^n\|_{L_a^p(\mathbb{C}, \phi)} \leq C_2 \left(\frac{pn+2-s}{s}\right)^{\frac{1}{2p}} \left(\frac{pn+2-s}{s \alpha}\right)^{\frac{2-s}{ps}} \|f\|_{L_a^p(\mathbb{C}, \phi)}.
\]
Using Lemma 5, we get the following lemma:

**Lemma 6.** The polynomial ring $C$ is dense in $L_a^p(\mathbb{C}, \phi)$.

Finally we show (ii)$\iff$(iii) in Theorem 1. Since every cyclic vector is non-vanishing, (iii)$\implies$(ii) is trivial. The idea for proving the opposite direction is from [Izu1].

**References**


