Circular Codes and Petri Nets

Genjiro Tanaka

Dept. of Computer Science, Shizuoka Institute of Science and Technology,
Fukuroi-shi, 437-8555 Japan

Abstract

The purpose of this paper is to investigate the relationship between limited codes and Petri nets. For a given Petri net with an initial marking $\mu$, we can naturally define an automaton $A$ which has the initial marking $\mu$ as an initial state, the reachability set $Re(\mu)$ as a set of states, and the set of transitions as a set of inputs. We can define prefix codes by considering the set of firing sequences which arrive from the positive initial marking of a Petri net to a certain subset of the reachability set[10,12]. The set $M$ of all positive firing sequences which start from the positive initial marking $\mu$ of a Petri net and reach $\mu$ itself forms a pure monoid. Our main interest is in the base $D$ of $M$. The family of pure monoids contains the family of very pure monoids, and the base of a very pure monoid is a circular code. Therefore, we can expect that $D$ may be a circular code. Here, for “small” Petri nets, we discuss under what conditions $D$ is circular.

Key words: Petri net, Code, Prefix code, Circular code, Limited code.

1. Introduction

Let $A$ be an alphabet, $A^*$ the free monoid over $A$, and $1$ the empty word. A word $v \in A^*$ is a left factor of a word $u \in A^*$ if there is a word $w \in A^*$ such that $u = vw$. The left factor $v$ of $u$ is called proper if $v \neq u$. A right factor and a proper right factor of a word are defined in a symmetric manner.

For a word $w \in A^*$ and a letter $x \in A$ we let $|w|_x$ denote the number of $x$ in $w$. The length of $w$ is the number of letters in $w$. A non-empty subset $C$ of $A^+$ is said to be a code if for $x_1, \ldots, x_p, y_1, \ldots, y_q \in C$, $p, q \geq 1$,

$$x_1 \cdots x_p = y_1 \cdots y_q \text{ implies } p = q \text{ and } x_1 = y_1, \ldots, x_p = y_p.$$ 

A subset $M$ of $A^*$ is a submonoid of $A^*$ if $M^2 \subseteq M$ and $1 \in M$. Every submonoid $M$ of a free monoid has a unique minimal set of generators

$$C = (M - \{1\}) - (M - \{1\})^2.$$ 

$C$ is called the base of $M$.

This is the abstract and the details will be published elsewhere.
A submonoid $M$ is **right unitary** in $A^*$ if for all $u, v \in A^*$,

$$u, uv \in M \implies v \in M.$$

$M$ is called **left unitary** in $A^*$ if it satisfies the dual condition. A submonoid $M$ is **biunitary** if it is both left and right unitary.

**Definition 1.1.** Let $M$ be a submonoid of a free monoid $A^*$, and $C$ its base. If $CA^* \cap C = \emptyset$, (resp. $A^*C \cap C = \emptyset$), then $C$ is called a *prefix* (resp. *suffix*) code over $A$. $C$ is called a *bifix* code if it is a prefix and suffix code.

A submonoid $M$ of $A^*$ is right unitary (resp. biunitary) if and only if its minimal set of generator is a prefix code (bifix code) ([1,p.46]).

**Definition 1.2.** A Petri net is a 5-tuple, $PN = (P, A, F, W, \mu_0)$ where:

$P = \{p_1, p_2, \ldots, p_n\}$ is a finite set of places,

$A = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions,

$F \subseteq (P \times A) \cup (A \times P)$ is a set of arcs,

$W : F \rightarrow \{1, 2, \ldots\}$ is a weight function,

$\mu_0 : P \rightarrow \{0, 1, 2, \ldots\}$ is the initial marking,

$P \cap A = \emptyset$ and $P \cup A \neq \emptyset$.

We use the following notations for a pre-set and a post-set:

$t = \{p : (p, t) \in F\}, \quad t\cdot = \{p : (t, p) \in F\},$

In this paper we shall assume that a Petri net has no isolated transitions, i.e., no $t$ such that $t \cup t\cdot = \emptyset$. A marking $\mu_0$ can be represented by a vector:

$$\mu_0 = (\mu_0(p_1), \mu_0(p_2), \ldots, \mu_0(p_n)), \quad p_i \in P, \quad n = |P|.$$

For every $t \in A$ the vector $\Delta t$ is defined by

$$\Delta t = (\Delta t(p_1), \Delta t(p_2), \ldots, \Delta t(p_n)), \quad n = |P|,$$

where

$$\Delta t(p) = \begin{cases} 
-W(p, t) + W(t, p) & \text{if } p \in t \cap t\cdot, \\
-W(p, t) & \text{if } p \in t - t\cdot, \\
W(t, p) & \text{if } p \in t - t\cdot, \\
0 & \text{if } p \notin t \cup t\cdot. 
\end{cases}$$

A transition $t \in A$ is said to be enabled in $\mu_0$, if $W(p, t) \leq \mu_0(p)$ for all $p \in t$. A firing of an enabled transition $t$ removes $W(p_1, t)$ tokens from each input place $p_1 \in t$, and adds $W(t, p_2)$ tokens to each output place $p_2 \in t$. Firing of an enabled transition $t$ at $\mu_0$ produces a new
marking $\mu_1$ such that

$$
\mu_1(p) = \begin{cases} 
\mu_0(p) - W(p, t) & \text{if } p \in t \cdot t, \\
\mu_0(p) + W(t, p) & \text{if } p \in t \cdot t, \\
\mu_0(p) - W(p, t) + W(t, p) & \text{if } p \in t \cap t, \\
\mu_0(p) & \text{otherwise.}
\end{cases}
$$

If we obtain the marking $\mu'$ that results from a firing of $t$ at $\mu$, we write $\delta(\mu, t) = \mu'$. A word $w = t_1t_2\ldots t_r$, $(t_i \in A)$, of transitions is said to be a (firing) sequence from $\mu_0$ if there exist markings $\mu_i, 1 \leq i \leq r$, such that $\delta(\mu_{i-1}, t_i) = \mu_i$ for all $i$, $(1 \leq i \leq r)$. In this case, $\mu_r$ is reachable from $\mu_0$ by $w$ and we write $\delta(\mu_0, w) = \mu_r$. The set of all possible markings reachable from $\mu_0$ is denoted by $\text{Re}(\mu_0)$, and the set of all possible sequences from $\mu_0$ is denoted by $\text{Seq}(\mu_0)$. The function $\delta : \text{Re}(\mu_0) \times T \rightarrow \text{Re}(\mu_0)$ is called a next-state function of a Petri net $PN$ [7,p.23]. We note that the above condition for $r = 0$ is understood to be $\mu_0 \in \text{Re}(\mu_0)$.

A marking $\mu$ is said to be positive if $\mu(p) > 0$ for all $p \in P$. A sequence $t_1t_2\ldots t_n \in \text{Seq}(\mu_0)$, $(t_i \in T)$, is called a positive sequence from $\mu_0$ if $\delta(\mu_0, t_1t_2\ldots t_i)$ is positive for all $i$, $(1 \leq i \leq n)$. The set of all positive sequences from $\mu_0$ is denoted by $\text{PSeq}(\mu_0)$. By $\text{PRe}(\mu_0)$ we denote the set of all possible positive markings reachable from $\mu_0$; $\text{PRe}(\mu_0) = \{\delta(\mu_0, w) | w \in \text{PSeq}(\mu_0)\}$.

2. Some codes related to Petri nets

For a Petri net $PN = (P, T, F, W, \mu_0)$ and a subset $X \subseteq \text{Re}(\mu_0)$ we can define a deterministic automaton $A(PN)$ as follows: $\text{Re}(\mu_0), T, \delta : \text{Re}(\mu_0) \times T \rightarrow \text{Re}(\mu_0), \mu_0, \text{ and } X$, are regarded as a state set, an input set, a next-state function, an initial state, and a final set of $A(PN)$, respectively. By using such automata, in [10,12] we defined four kinds of prefix codes and examined fundamental properties of these codes.

Let $PN = (P, A, F, W, \mu)$ be a Petri net. The set

$$
\text{Stab}(PN) = \{\mu | w \in \text{Seq}(\mu) \text{ and } \delta(\mu, w) = \mu\}
$$

forms a submonoid of $A^*$. If $\text{Stab}(PN) \neq \{1\}$, then we denote the base of $\text{Stab}(PN)$ by $S(PN)$. Since $S(PN)A^+ \cap S(PN) = \emptyset$, $S(PN)$ is a prefix code over $A$.

A submonoid $M$ of $A^*$ is called pure [7] if for all $x \in A^*$ and $n \geq 1$,

$$
x^n \in M \implies x \in M.
$$

A subsemigroup $H$ of a semigroup $S$ is extractable in $S$ [9,p.191] if

$$
x, y \in S, z \in H, xyz \in H \implies xyz \in H.
$$

Proposition 2.1. If $\text{Stab}(PN) \neq \emptyset$, then $\text{Stab}(PN)$ is a biunitary extractable pure monoid.
Definition 2.1. Let \( PN = (P, A, F, W, \mu) \) be a Petri net with a positive marking \( \mu \). Define the subset \( D(PN) \) as a set of all positive sequence \( w \) of \( S(PN) \).

Since \( D(PN) \) is a subset of \( S(PN) \), \( D(PN) \) is a bifix code over \( A \).

Proposition 2.2. If \( D(PN) \neq \emptyset \), then \( D(PN)^* \) is a biunitary extractable pure monoid.

Example 2.1. Let \( PN = ((p, q), \{a, b\}, F, W, \mu_0) \) be a Petri net defined by \( W(a, p) = W(p, b) = W(q, a) = W(b, q) = 1 \), \( \mu_0(p) = \mu_0(q) = 2 \). Then \( D(PN) = \{ab, ba\} \), therefore \( \{ab, ba\}^* \) is pure [1, p.324, Ex.1.3].

Proposition 2.3. If \( z, xzy \in D(PN), x, y \in A^+ \), then \( xz^*y \in D(PN) \).

A code \( D \) is infix if \( w, xwy \in D \) implies \( x = y = 1 \) [8, p.129].

Proposition 2.4. If \( D(PN) \) is a non-empty finite set, then \( D(PN) \) is an infix code.

3. Limited code

A submonoid \( M \) of \( A^* \) is very pure if for all \( u, v \in A^* \),
\[
    u, v \in A^*, uv, vu \in M \Rightarrow u, v \in M.
\]

The base of a very pure monoid is called a circular code.

Let \( p, q \geq 0 \) be two integers. If for any sequence \( u_0, u_1, ..., u_{p+q} \) of words in \( A^* \), the assumptions \( u_{i-1}u_i \in M \) \( (1 \leq i \leq p + q) \) imply \( u_p \in M \), then a submonoid \( M \) is said to satisfy condition \( C(p, q) \). If a submonoid \( M \) of \( A^* \) satisfies condition \( C(p, q) \), then \( M \) is very pure [1, p.329, Proposition 2.1], and its base is called a \((p, q)-\text{limited code}\).

If a submonoid \( D \) of \( A^* \) is a bifix \((1,1)\)-limited code, then for any \( u_0, u_1, u_2 \in A^* \) such that \( u_0u_1, u_1u_2 \in D \) we have \( u_1 \in D \). Thus \( u_0u_1, u_1, u_2 \in D \). This implies that \( u_0, u_1, u_2 \in D \), since \( D \) is bifix. Therefore \( D \) is \((2,0)-(1,1)-\) and \((0,2)-\)limited.

Let \( PN_0 = ((p), \{a, b\}, F, W, \mu_0) \) be a Petri net such that \( W(a, p) = \alpha, W(p, b) = \beta, \mu_0 = (\lambda_p), \lambda_p > 0 \).
Consider the set $\Omega$ of positive markings in $PN_0$

$$\Omega = \{ \mu | \mu = \mu_0 + \Delta(w), w \in PSeq(\mu_0) \}.$$

$\alpha$ and $\beta$, and let $N = \{0, 1, 2, \cdots\}$ be a set of non-negative integers. Then we have

(0) $D(PN_0)$ is dense.

(1) If $\lambda_p < g$, then $\Omega = \{\lambda_p + ng | n \in N\}$.

(2) If $\lambda_p = sg, s \geq 0, s \in N$, then $\Omega = \{ng | n \geq 1, n \in N\}$.

(3) If $\lambda_p = sg + t_p, s \geq 0, s \in N, 0 < t_p < g$, then $\Omega = \{t_p + ng | n \geq 0, n \in N\}$.

**Proposition 3.1.** If $\lambda_p > \gcd(\alpha, \beta)$, then $D(PN_0)$ is not circular.

**Proposition 3.2.** $D(PN_0)$ is circular if and only if $\lambda_p \leq \gcd(\alpha, \beta)$.

Let $PN_1 = (\{p, q\}, \{a, b\}, F, W, \mu_0)$ be a Petri net such that $W(a, p) = \alpha, W(p, b) = \alpha', W(q, a) = \beta, W(b, q) = \beta', \mu_0(p) = \lambda_p, \mu_0(q) = \lambda_q$.

Suppose that $D(PN_1) \neq \emptyset$ and $w \in D(PN_1)$. Let $n = |w|_a$ and $m = |w|_b$, then $\Delta(w) = n\Delta(a) + m\Delta(b) = 0$ (zero vector). Consequently the linear equation

$$\begin{pmatrix} \alpha & -\alpha' \\ -\beta & \beta' \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a non-trivial solution in $N$. Thus $\alpha\beta' = \alpha'\beta$. Therefore, if $D(PN_1) \neq \emptyset$, then $PN_1 = (\{p, q\}, \{a, b\}, F, W, \mu_0)$ has the following forms:

$W(a, p) = \alpha, W(p, b) = k\alpha, W(q, a) = \beta, W(b, q) = k\beta, k > 0$.

Here we assume that $k$ is an integer. That is, we define a Petri net $PN_1 = (\{p, q\}, \{a, b\}, F, W, \mu_0)$ as follows

$$\Delta(a) = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}, \quad \Delta(b) = \begin{pmatrix} -k\alpha \\ k\beta \end{pmatrix},$$

where $k$ is a positive integer.

We define an integer $M_p$ as follows

$$M_p = \begin{cases} \frac{\lambda_p}{\alpha} - 1, & \text{if } \frac{\lambda_p}{\alpha} \text{ is an integer}, \\ \lceil \frac{\lambda_p}{\alpha} \rceil, & \text{if } \frac{\lambda_p}{\alpha} \text{ is not an integer}. \end{cases}$$
where \([\cdot]\) is the symbol of Gauss. Similarly we define an integer \(M_q\) as follows, \(M_q = \lfloor \frac{x}{y} \rfloor - 1\) if \(\frac{x}{y}\) is an integer, and \(M_q = \lfloor \frac{x}{y} \rfloor\) if \(\frac{x}{y}\) is not an integer.

**Proposition 3.3.** We have

1. If \(M_p + M_q > k, M_p \geq k\) and \(M_q \geq 1\), then \(D(PN_1)\) is not circular.
2. If \(M_p + M_q > k, k > M_p \geq 1, M_p + M_q > k\), then \(D(PN_1)\) is not circular.
3. If \(M_p + M_q = k, M_p \geq 1, M_q \geq 1\), then \(D(PN_1)\) is a singleton.
4. If \(M_p + M_q \geq k, M_p = 0, M_q \geq 0\), then \(D(PN_1)\) is \((1,1)\)-limited.

**Corollary 3.1.** Let \(n\) and \(k\) be arbitrary integers such that \(n > k > 1\). Define the automaton

\[ A_{(n,k)} = (\{1, 2, \ldots, n\}, \{a, b\}, f, 1, \{1\}) \]

by \(f(i, a) = i + 1, 1 \leq i \leq n - 1\) and \(f(j, b) = j - k, k + 1 \leq j \leq n\). Then the base of language \(L(A_{(n,k)})\) recognized by \(A_{(n,k)}\) is a \((1,1)\)-limited code.

**Proposition 3.4.** Let \(PN = (\{p_1, \ldots, p_n\}, \{a_1, \ldots, a_n\}, F, W, \mu_0), n \geq 2\), be a Petri net such that \(W(p_i, a_i) = \alpha_i, W(a_i, p_{i+1}) = \beta_i, 1 \leq i \leq n - 1\), and \(W(p_n, a_n) = \alpha_n, W(a_n, p_1) = \beta_n\). \(\mu_0 = (\lambda_1, \ldots, \lambda_n), \mu_0(p_1) = \lambda_{i_1}, 1 \leq i \leq n\). Furthermore let \(g_j = \text{gcd}(\beta_{j-1}, \alpha_j), 2 \leq j \leq n\). If \(\lambda_1/\alpha_1 > 1\) and \(\lambda_i \leq g_i\), for all \(i = 2, \ldots, n\), then \(D(PN)\) is \((1,1)\)-limited.

**Lemma 3.1.** Let \(PN_2\) be a Petri net mentioned above, and let \(\alpha = \text{gcd}(\alpha_1, \alpha_2, \alpha_3), \beta = \text{gcd}(\beta_1, \beta_2)\). Suppose that \(D(PN_2) \neq \emptyset\). If \(d \in D(PN_2)\) and \(v\) is its proper suffix, then we have one of the following:

1. \(\Delta(v)(p_1) \leq -\alpha, \Delta(v)(p_2) \leq -\beta\).
2. \(\Delta(v)(p_1) = 0, \Delta(v)(p_2) \leq -\beta\).
3. \(\Delta(v)(p_1) \leq -\alpha, \Delta(v)(p_2) \leq 0\).

**Proposition 3.5.** If \(D(PN_2) \neq \emptyset\) and \(\lambda_1 \leq \alpha, \lambda_2 \leq \beta\), then \(D(PN_2)\) is \((1,1)\)-limited.

Let \(PN_3 = (\{p, q\}, \{a, b, c\}, W, \mu_0)\) be a Petri net such that \(W(a, p) = \alpha, W(q, a) = \beta, W(p, b) = \)
\[ \alpha + \beta, W(b, q) = \alpha + \beta, W(c, p) = \beta, W(q, c) = \alpha, \mu_0(p) = \lambda_p, \mu_0(q) = \lambda_q. \]

**Lemma 3.2.** Let \( PN_3 \) be a Petri net mentioned above. Suppose that \( \beta < \lambda_p \leq \alpha + \beta \) and \( \beta < \lambda_q \leq \alpha \), then for any \( u \in PSeq(PN_3) \) we have one of the following.

1. \( \Delta(u) = \begin{pmatrix} k(\alpha - \beta) \\ k(\alpha - \beta) \end{pmatrix}, \) \( k \geq 0, \)
2. \( \Delta(u) = \begin{pmatrix} k(\alpha - \beta) + l\alpha \\ k(\alpha - \beta) - l\beta \end{pmatrix}, \) \( k \geq 0, l \geq 1, \)
3. \( \Delta(u) = \begin{pmatrix} k(\alpha - \beta) - l\beta \\ k(\alpha - \beta) + l\alpha \end{pmatrix}, \) \( k \geq 0, l \geq 1. \)

**Proposition 3.6.** Suppose that \( D(PN_3) \neq \emptyset \). If \( \beta < \lambda_p \leq \alpha + \beta \) and \( \beta < \lambda_q \leq \alpha \), then \( D(PN_3) \) is \((1,1)\)-limited.

**References**