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Rate of convergence of the Bence-Merriman-Osher algorithm for motion by mean curvature

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1 Introduction

This is a brief report of [16].

In 1992, Bence, Merriman and Osher proposed in [2] an algorithm for computing the motion of a hypersurface by its mean curvature. It is described as follows.

Let $C_0 \subset \mathbb{R}^N$ be a closed set and let $u = u(t, x)$ be the solution of

$$(1.1) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = \begin{cases} 1 & x \in C_0, \\ -1 & x \in \mathbb{R}^N \setminus C_0. \end{cases} \end{cases}$$

Fix a time step $h > 0$ and set

$$C_1 = \{x \in \mathbb{R}^N \mid u(h, x) \geq 0\}.$$

Next we solve (1.1) with C_0 replacing C_1 and define a new set C_2 with u replaced by the solution of (1.1) with the new initial data. Repeating this procedure, we have a sequence $\{C_k\}_{k=0,1,\ldots}$ of closed sets in $\mathbb{R}^N$. Then we define

$$C_t^h = C_k \text{ if } kh \leq t < (k+1)h, \ k = 0, 1, \ldots$$

for $t \geq 0$. Letting $h \to 0$, we obtain in the limit a flow $\{D_t\}_{t \geq 0}$ of closed subsets in $\mathbb{R}^N$ with $D_0 = C_0$ and then $\partial D_t (:= \Gamma_t)$ moves by its $((N-1)$-times) mean curvature. That is, Γ_t satisfies

$$V = \kappa \quad \text{on } \Gamma_t, t > 0.$$

Here $V = V(t, x)$ is the normal velocity of Γ_t at $x \in \Gamma_t$ and $\kappa = \kappa(t, x)$ is the mean curvature of Γ_t at $x \in \Gamma_t$.

The convergence of the Bence-Merriman-Osher (BMO) algorithm was proved by Mascarenhas [19], Evans [5], Barles-Georgelin [1] and Goto-Ishii-Ogawa [9]. The generalizations of the BMO algorithm were considered by Ishii [13], Ishii-Pires-Souganidis [15], Ishii-Ishii [14], Vivier [22] and Leoni [18]. However, to the author's knowledge, there are a few results on the rate of convergence of the BMO algorithm. In [21] Ruuth gave a time-local error estimate of the BMO algorithm in $\mathbb{R}^2$. Nakamura and the author proved in [17] the Hausdorff distance between the motion by mean curvature Γ_t and the approximate interface $\Gamma_t^h = \partial C_t^h$ is of order $h^{1/2}$. This result is valid before the onset of singularities, but not optimal.

The purposes of this article are to present the rate of convergence of the BMO algorithm globally in time and to show its optimality. In fact, assuming $\{\Gamma_t\}_{0\leq t<T_0}$ is the motion of a smooth and compact hypersurface by mean curvature, we prove that, for any $T < T_0$,

$$\sup_{t\in[0,T]} d_H(\Gamma_t, \Gamma_t^h) \leq Lh,$$

where L is a constant depending on T, but independent of small $h > 0$ and d_H denotes the Hausdorff distance. We can show that this is the optimal rate in the case of a circle evolving by curvature.

Both of the order in h and the optimality are the consequence of the maximum principle and the explicit constructions of sub- and supersolutions of (1.1), which is inspired by the asymptotic analysis of solutions of the Allen-Cahn equation (see, e.g., Fife [6] and de Mottoni-Schatzman [4]). As for the relation between the BMO algorithm and the Allen-Cahn equation, from the viewpoint of the splitting methods in numerical analysis, Vivier [22] first pointed out that we may think the Allen-Cahn equation is an approximation of the BMO algorithm. Leoni [18] and Goto-Ishii-Ogawa [9] gave the proofs of the convergence of the BMO algorithm and a generalized scheme by applying some techniques of the asymptotic analysis for the Allen-Cahn equation. The arguments in this paper also rely on them.

This article is organized as follows. In section 2 we state our main results. Section 3 is devoted to the formal asymptotic expansion of the solution of (1.1). In section 4 we give the outline of the proof of Theorem 2.1 below.

2 Main results

First, we state the rate of convergence. For this purpose, we rewrite the BMO algorithm as follows: Let $\Gamma_0 \subset \mathbb{R}^N$ be a smooth and compact hypersurface and $C_0 \subset \mathbb{R}^N$ the compact set such that $\partial C_0 = \Gamma_0$. Fix a time step $h > 0$. Let $u^h = u^h(t,x)$ be the solution of

$$\begin{cases} u_t^h = \Delta u^h \quad \text{in } (kh,(k+1)h)\times\mathbb{R}^N, \\ u^h(kh,x) = \begin{cases} 1, & x\in C_k, \\ -1, & x\in \mathbb{R}^N\setminus C_k, \end{cases} \\ C_k = \begin{cases} \text{the given set } C_0 & \text{for } k=0, \\ \left\{x\in\mathbb{R}^N \;\middle|\; \lim\limits_{t\to kh-} u^h(t,x) \geq 0\right\} & \text{for } k=1,2,\ldots \end{cases} \end{cases} \tag{2.1}$$

Set

$$\Gamma_t^h = \begin{cases} \{x\in\mathbb{R}^N \mid u^h(t,x)=0\} & \text{for } t\neq kh, \\ \partial C_k & \text{for } t=kh. \end{cases} \tag{2.2}$$

We note that Γ_t^h is a smooth and compact hypersurface for each $t \geq 0$, $h > 0$. Using this formulation, we have the following theorem.

Theorem 2.1 *Let $\{\Gamma_t\}_{0 \leq t < T_0}$ be the motion of a smooth and compact hypersurface by mean curvature starting from Γ_0. Let Γ_t^h be defined by (2.2). Then, for any $T \in (0, T_0)$, there exist $h_0 > 0$ and $L > 0$ such that*

$$\sup_{t \in [0,T]} d_H\left(\Gamma_t^h, \Gamma_t\right) \leq Lh \tag{2.3}$$

for all $h \in (0, h_0)$. Here $d_H(A, B)$ denotes the Hausdorff distance between A, $B \subset \mathbb{R}^N$.

Remark 2.1 (1) As to the motion by mean curvature $\{\Gamma_t\}_{0 \leq t < T_0}$, the following results are well known: Assume that Γ_0 is the smooth boundary of a bounded domain. If $N = 2$ or Γ_0 is convex, then Γ_t evolves smoothly and it shrinks to a point at a finite time. (see Gage - Hamilton [7], Grayson [10] and Huisken [12]). In other cases the singularities may appear before Γ_t shrinks to a point (see, e.g., Grayson [11]). Therefore (2.3) is valid before Γ_t shrinks to a point or develops the singularities.
(2) We can formally derive (2.3) by using the result of Ruuth [21]. However, it seems that its mathematical proof has not yet given.

The (2.3) is optimal with respect to the order of h. To see this, we consider the following situation: Set $\Gamma_0 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ and $\Gamma_0 = \{x \in \mathbb{R}^2 \mid |x| = \phi(t)\}$ ($\phi(t) = \sqrt{1 - 2t}$). Then it is easily seen that Γ_t moves by its curvature and it shrinks to the origin at $T_{max} = 1/2$. For each $h > 0$, let $\{\Gamma_t^h\}_{t \geq 0}$ be the flow defined by (2.2) satisfying $\Gamma_0^h = \Gamma_0$. We note that, for each t, $h > 0$, Γ_t^h is also a circle centered at origin and thus we can define its radius $R_h(t)$. In this setting, we obtain the following estimate.

Theorem 2.2 *For each $T \in (0, T_{max})$, there exist $h_0 > 0$ and $L > 0$ such that*

$$\sup_{t \in [0,T]} |R_h(t) - (\phi(t) - h\psi(t))| \leq Lh^{3/2} \tag{2.4}$$

for all $h \in (0, h_0)$. Here $\psi(t) = -\dfrac{\log \phi(t)}{3\phi(t)}$.

This theorem implies the optimality of (2.3). In addition, we see that Γ_t^h moves faster than Γ_t.

3 Formal asymptotic expansion

To prove Theorem 2.1 and 2.2, we construct suitable sub- and supersolutions of (1.1). For this purpose, we apply the method of asymptotic expansion of solutions of Allen-Cahn equation. As for this method, see Fife [6], de Mottoni-Schatzman [4] etc.

Let u be the solution of (1.1). Set $\widetilde{\Gamma}_t = \{x \in \mathbb{R}^N \mid u(t,x) = 0\}$. Since it follows from Goto-Ishii-Ogawa [9, Proposition 6.1] that $u(t,\cdot) \simeq 1$ or -1 away from $\widetilde{\Gamma}_t$ as $t \searrow 0$, we have only to consider the asympotics of u near $\widetilde{\Gamma}_t$. We define the signed distance function $\widetilde{d}$ to $\widetilde{\Gamma}_t$ as follows:

$$\widetilde{d}(t,x) = \begin{cases} \operatorname{dist}(x, \widetilde{\Gamma}_t) & \text{for } x \in \widetilde{P}_t, \\ -\operatorname{dist}(x, \widetilde{\Gamma}_t) & \text{for } x \in \mathbb{R}^N \backslash \widetilde{P}_t, \end{cases}$$

where $\widetilde{P}_t$ is the domain enclosed by $\widetilde{\Gamma}_t$. By Evans [5, Theorem 4.1] we may consider that $\widetilde{\Gamma}_t$ moves by mean curvature. Then we see that, for some K, $\delta > 0$, $\widetilde{d}$ satisfies

$$(3.1) \qquad |\widetilde{d}_t - \Delta\widetilde{d} - \kappa^s\widetilde{d}| \le K\widetilde{d}^2 \quad \text{for } 0 < t < h,\ x \in \{y \in \mathbb{R}^N \mid |\widetilde{d}(t,y)| < \delta\},$$

where $\kappa^s = \kappa^s(t,x)$ is the sum of squares of all principal curvatures of $\widetilde{\Gamma}_t$ at $x \in \widetilde{\Gamma}_t$ (see, e.g., Chen [3], Gilbarg-Trudinger [8] and Paolini-Verdi [20]).

Assume that u can be expanded near $\widetilde{\Gamma}_t$ as follows:

$$u(t,x) = V_0\left(t, x, \frac{\widetilde{d}(t,x)}{2\sqrt{t}}\right) + \sqrt{t}V_1\left(t, x, \frac{\widetilde{d}(t,x)}{2\sqrt{t}}\right) + tV_2\left(t, x, \frac{\widetilde{d}(t,x)}{2\sqrt{t}}\right) + \cdots.$$

We introduce a new variable $\rho = \widetilde{d}(t,x)/2\sqrt{t}$. Substituting the right-hand side of this equality into (1.1), we get the following:

$$\begin{aligned} 0 \;=\; & -\frac{1}{4t}(V_{0,\rho\rho} + 2\rho V_{0,\rho}) \\ & -\frac{1}{\sqrt{t}}\left\{V_{0,\rho}\langle DV_0, Dd\rangle + \left(\frac{1}{4}V_{0,\rho\rho} + \frac{\rho}{2}V_{1,\rho} - \frac{1}{2}V_1\right)\right\} \\ & +\left\{V_{0,t} - \Delta V_0 + \kappa^s\rho V_{0,\rho} + \frac{V_{1,\rho}}{2}(d_t - \Delta d - 2\langle DV_1, Dd\rangle)\right. \\ & \qquad \left. - \left(\frac{1}{4}V_{2,\rho\rho} + \frac{\rho}{2}V_{2,\rho} - V_2\right)\right\} \\ & + \cdots, \end{aligned}$$

where $DV_j = (\partial V_j/\partial x_1, \ldots, \partial V_j/\partial x_N)$, $V_{j,\rho} = \partial V_j/\partial\rho$, $V_{j,\rho\rho} = \partial^2 V_j/\partial\rho^2$ $(j = 0, 1, 2, \ldots)$ and we have used (3.1).

We compare the coefficients of $t^{j/2}$ $(j = -2, -1, 0, 1, \ldots)$. In the case of the t^{-1}-term, we have

$$V_{0,\rho\rho} + 2\rho V_{0,\rho} = 0 \quad \text{on } \mathbb{R}.$$

Taking into the account Goto-Ishii-Ogawa [9, Proposition 6.1], we impose the following condition on V_0:

$$V_0(t,x,\rho) \longrightarrow \begin{cases} 1, & (\rho \to +\infty), \\ -1, & (\rho \to -\infty). \end{cases}$$

Hence we obtain $V_0(t,x,\rho) = V_0(\rho) = \frac{2}{\sqrt{\pi}} \int_0^{\rho} e^{-s^2} ds$. As for the $t^{-1/2}$-term, we get

$$\frac{1}{4} V_{0,\rho\rho} + \frac{\rho}{2} V_{1,\rho} - \frac{1}{2} V_1 = 0 \quad \text{on } \mathbb{R}.$$

Since $u - V_0$ converges to 0 as $\rho \to \pm\infty$ exponentially, we see that V_1 satisfies

$$V_1(t,x,\rho) \longrightarrow 0 \quad (\rho \to \pm\infty).$$

Then $V_1 \equiv 0$. By a similar way, we observe that V_2 satisfies

$$\frac{1}{4} V_{2,\rho\rho} + \frac{\rho}{2} V_{2,\rho} - V_2 = \kappa^s \rho V_{0,\rho} \quad \text{on } \mathbb{R}, \ V_2(t,x,\rho) \longrightarrow 0 \quad (\rho \to \pm\infty).$$

Thus we obtain $V_2 = -\frac{\kappa^s \rho}{2} V_{0,\rho}$.

Continuing the above processes, we can obtain the formal asymptotic expansion of u.

4 Outline of the proof of Theorem 2.1

We mention only the outline of the proof of Theorem 2.1 because the proof of Theorem 2.2 is similar to that of Theorem 2.1, but it is more complicated.

Let $\{\Gamma_t\}_{0 \le t < T_0}$ be the motion of a smooth and compact hypersurface by mean curvature. We define the signed distance function $d(t,x)$ to Γ_t by

$$d(t,x) = \begin{cases} \operatorname{dist}(x, \Gamma_t) & \text{for } x \in P_t, \\ -\operatorname{dist}(x, \Gamma_t) & \text{for } x \in \mathbb{R}^N \setminus P_t, \end{cases}$$

where P_t is the bounded domain enclosed by Γ_t. Fix $T \in (0, T_0)$ and $h > 0$. Set $m = [T/h]$. Let u^h be the solution of (2.1) and let $\{\Gamma_t^h\}_{t,h}$ be defined by (2.2). Then, for each t, $h > 0$, Γ_t^h is a smooth and compact hypersurface.

According to the formal asymptotic expansion, we define

$$v^{0,\pm}(t,x) = V_0\left(\frac{d(t,x)}{2\sqrt{t}}\right) + tV_2\left(t, x, \frac{d(t,x)}{2\sqrt{t}}\right) \pm t^{3/2} V_3.$$

Here V_0 and V_2 is the functions obtained in the previous section. At $t = 0$, we set

$$v^{0,-}(0,x) = v^{0,+}(0,x) = u^h(0,x).$$

Taking $V_3 > 0$ sufficiently large and independent of $h > 0$, we can show that $v^{0,-}$, $v^{0,+}$ are, respectively, a subsolution and a supersolution of (1.1) in $(0,h) \times \mathbb{R}^N$. Hence, by the comparison principle for the heat equation, we get

$$v^{0,-} \le u^h \le v^{0,+} \quad \text{on } [0,h] \times \mathbb{R}^N. \tag{4.1}$$

In addition, it is easily seen from (4.1) and the definitions of $v^{0,\pm}$ that

(4.2) $$v^{0,-}(t,x) \le 0 \le v^{0,+}(t,x) \quad \text{for } t \in (0,h],\ x \in \Gamma_t^h,$$

(4.3) $$v^{0,+}(t,x) = t^{3/2}V_3,\ v^{0,-}(t,x) = -t^{3/2}V_3 \quad \text{for } t \in (0,h],\ x \in \Gamma_t,$$

(4.4) $$v^{0,+}(t,x) < 0 \text{ for } t \in (0,h],\ x \in \{y \in \mathbb{R}^N \mid d(t,y) < -\sqrt{t}\},$$

(4.5) $$v^{0,-}(t,x) > 0 \text{ for } t \in (0,h],\ x \in \{y \in \mathbb{R}^N \mid d(t,y) > \sqrt{t}\},$$

(4.6) $$D_n v^{0,\pm}(t,x) \ge \frac{K}{\sqrt{t}} \quad \text{for } t \in (0,h],\ x \in \Theta_t^h,$$

where $D_n v(t,x)$ is the derivative of v with respect to the inner normal direction to Γ_t^h at $x \in \Gamma_t^h$ and $K > 0$ is a constant independent of $h > 0$. Put

$$\Sigma_t^h := \{x \in \mathbb{R}^N \mid v^{0,-}(t,x) \le 0 \le v^{0,+}(t,x)\},\ \Theta_t^h := \{x \in \mathbb{R}^N \mid |d(t,x)| \le \sqrt{t}\},$$
$$P_t^h = \text{the bounded domain enclosed by } \Gamma_t^h.$$

Then, from (4.2) - (4.5), we have

(4.7) $$\Gamma_t \cup \Gamma_t^h \subset \Sigma_t^h \subset \Theta_t^h \quad \text{for } t \in [0,h].$$

Using (4.3), (4.6) and (4.7), we observe that

(4.8) $$\Gamma_t \cup \Gamma_t^h \subset \Sigma_t^h \subset \{x \in \mathbb{R}^N \mid |d(t,x)| \le C_1 t^2\} \quad \text{for } t \in [0,h],$$

where $C_1 > 0$ is a large constant independent of $h > 0$. Let $\Gamma_t^{h,+} = \partial(P_t^h \cap P_t)$ and $\Gamma_t^{h,-} = \partial(P_t^h \cup P_t)$. Then $\Gamma_t^{h,+} \cup \Gamma_t^{h,-} = \Gamma_t^h \cup \Gamma_t$ and thus, by (4.8) and some elementary arguments, we obtain

$$\begin{aligned} d_H(\Gamma_t^h, \Gamma_t) &= \max\{d_H(\Gamma_t^{h,+}, \Gamma_t), d_H(\Gamma_t^{h,-}, \Gamma_t)\} \le \sup_{x:|d(t,x)|=C_1t^2} \operatorname{dist}(x, \Gamma_t) \\ &= C_1 t^2 \le C_1 h^2 \quad \text{for } t \in [0,h]. \end{aligned}$$

Set $\alpha_1 = C_1$ and define

$$v^{1,\pm}(t,x) = V_0\left(\frac{d^{1,\pm}(t,x)}{2\sqrt{t}}\right) + tV_2\left(t,x,\frac{d^{1,\pm}(t,x)}{2\sqrt{t}}\right) \pm ((t-h)^{3/2}V_3 + V_4\alpha_1 h^2\sqrt{t-h}),$$
$$v^{1,\pm}(h,x) = \begin{cases} 1 & \text{on } \{x \in \mathbb{R}^N \mid d^{1,\pm}(h,x) \ge 0\}, \\ -1 & \text{in } \{x \in \mathbb{R}^N \mid d^{1,\pm}(h,x) < 0\}, \end{cases}$$
$$d^{1,\pm}(t,x) = d(t,x) \pm \alpha_1 h^2 \quad \text{for } t \in [h,2h].$$

Here V_3 is the same constant as above and $V_4 > 0$ is chosen sufficiently large and independent of $h > 0$. Then we can verify that $v^{1,-}$ and $v^{1,+}$ are, respectively, a subsolution and a supersolution of (2.1) in $(h,2h) \times \mathbb{R}^N$ satisfying $v^{1,-}(h,x) \le u^h(h,x) \le v^{1,+}(h,x)$ on $\mathbb{R}^N$. Hence we get, by the comparison principle for the heat equation,

$$v^{1,-} \le u^h \le v^{1,+} \quad \text{on } [h,2h] \times \mathbb{R}^N.$$

It is observed by a similar argument to the above that

$$d_H(\Gamma_t^h, \Gamma_t) \le \alpha_2 h^2 \quad \text{for } t \in [h, 2h] \quad (\alpha_2 := C_1 + (1 + C_2 h)\alpha_1)$$

for some $C_2 > 0$ independent of $h > 0$.

We repeat this argument inductively. Set $\alpha_k = C_1 + (1 + C_2 h)\alpha_{k-1}$ $(k \ge 2)$ and define

$$\begin{aligned}
v^{k,\pm}(t,x) &= V_0\left(\frac{d^{k,\pm}(t,x)}{2\sqrt{t}}\right) + tV_2\left(t, x, \frac{d^{k,\pm}(t,x)}{2\sqrt{t}}\right) \pm ((t-kh)^{3/2}V_3 + V_4\alpha_k h^2\sqrt{t-kh}),\\
v^{k,\pm}(kh,x) &= \begin{cases} 1 & \text{on } \{x \in \mathbb{R}^N \mid d^{k,\pm}(kh,x) \ge 0\},\\ -1 & \text{in } \{x \in \mathbb{R}^N \mid d^{k,\pm}(kh,x) < 0\},\end{cases}\\
d^{k,\pm}(t,x) &= d(t,x) \pm \alpha_k h^2 \quad \text{for } t \in [kh, (k+1)h].
\end{aligned}$$

We can check that $v^{k,-}$ and $v^{k,+}$ are, respectively, a subsolution and a supersolution of (2.1) in $(kh, (k+1)h) \times \mathbb{R}^N$ satisfying $v^{k,-}(kh,x) \le u^h(kh,x) \le v^{k,+}(kh,x)$ on $\mathbb{R}^N$. Applying the comparison principle for the heat equation, we get

$$v^{k,-} \le u^h \le v^{k,+} \quad \text{on } [kh, (k+1)h] \times \mathbb{R}^N.$$

By a similar argument to the case $k = 0$, we have

$$d_H(\Gamma_t^h, \Gamma_t) \le \alpha_{k+1} h^2 \quad \text{for } t \in [kh, (k+1)h] \quad (\alpha_{k+1} := C_1 + (1 + C_2 h)\alpha_k).$$

Finally we estimate $\{\alpha_k\}_{1 \le k \le m}$. It follows from the definitions of m and $\{\alpha_k\}_{1 \le k \le m}$ that

$$\alpha_k = C_1 \sum_{l=1}^{k} (1 + C_2 h)^{l-1} \le C_1 \frac{(1 + C_2 h)^m - 1}{C_2 h} \le C_1 m (1 + C_2 h)^{m-1} \le \frac{C_1 T e^{C_2 T}}{h}.$$

Thus we obtain (2.3) with $L = C_1 T e^{C_2 T}$. □

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