<table>
<thead>
<tr>
<th>Title</th>
<th>The automorphism groups of certain commutant subalgebras of lattice vertex operator algebras (Algebraic combinatorics and the related areas of research)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Sakuma, Shinya</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2006), 1476: 176-181</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/48219">http://hdl.handle.net/2433/48219</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
The automorphism groups of certain commutant subalgebras of lattice vertex operator algebras

佐久間伸也 (Shinya Sakuma)

東京大学数理科学研究科・学振研究員 PD
(Graduate School of Mathematical Sciences. The university of Tokyo)

1 Introduction

An element $e$ of weight 2 of a vertex operator algebra $V$ is called an Ising vector if the vertex subalgebra generated by $e$ is isomorphic to the simple Virasoro VOA $L(\frac{1}{2},0)$ with central charge $\frac{1}{2}$. Any Ising vector $e$ defines an automorphism $\tau_e$ of $V$ with $\tau_e^2 = 1$ by using representation of $L(\frac{1}{2},0)$. In the case of the Moonshine VOA $V^\infty$, $\tau_e$ gives a $2A$-involution of the Monster simple group $M = \text{Aut}(V^\infty)$. An Ising vector $e$ is called $\sigma$-type if $\tau_e = 1$. An Ising vector $e$ of $\sigma$-type defines an automorphism $\sigma_e$ of $V$ with $\sigma_e^2 = 1$. It is known that if a set $E$ of Ising vectors of $\sigma$-type such that $\sigma_e(f) \in E$ for any $e, f \in E$, the subgroup of Aut($V$) generated by $\{\sigma_e | e \in E\}$ is 3-transposition group. Matsuo classified all 3-transposition groups defined by such a set $E$ of Ising vectors of $\sigma$-type.

Let $R$ be a root lattice. Let $V_{\sqrt{2}R}$ be the lattice vertex operator algebra associated to the lattice whose norm is twice of $R$'s and $V_{\sqrt{2}R}^+$ the fixed point subalgebra of the lattice VOA $V_{\sqrt{2}R}$ by the lift of $(-1)$-isometry on $R$. There are a lot of Ising vectors (of $\sigma$-type) and conformal vectors in $V_{\sqrt{2}R}^+$. We consider the commutant subalgebra $M_R$ of a conformal vector $\omega_R$ fixed by Aut($R$) in $V_{\sqrt{2}R}^+$. Then Aut($R$)/($-1$) acts on $M_R$ faithfully.

This talk is about the result obtained by a joint work with Ching Hung Lam of National Cheng Kung University in Taiwan and Hiroshi Yamauchi of the University of Tokyo. We study the classification of Ising vectors of $V_{\sqrt{2}R}^+$.
Then we apply our results to study commutant subalgebras $M_R$ related to root lattice $R$. We completely classify all Ising vectors in such commutant subalgebras. Moreover, we show that $M_R$ is generated by Ising vectors and determine their full automorphism groups.

2 Ising vectors and $\sigma$-involutions

An element $e \in V_2$ is a conformal vector with central charge $c \in \mathbb{C}$ if $L(n) := e_{n+1}, n \in \mathbb{Z}$ satisfy the Virasoro relation

$$[L(m), L(n)] = (m + n)L_{m-n} + \delta_{m+n,0} \frac{m^3 - m}{12} c$$

for $m, n \in \mathbb{Z}$. A conformal vector $e$ of a VOA $V$ with central charge $\frac{1}{2}$ is called an Ising vector if the subalgebra $\text{Vir}(e)$ generated by $e$ is isomorphic to the simple Virasoro VOA $L(\frac{1}{2}, 0)$ with central charge $\frac{1}{2}$. It is well-known that the Virasoro VOA $L(\frac{1}{2}, 0)$ is rational and has exactly three irreducible modules $L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{16})$.

Let $e$ be an Ising vector of a VOA $V$. Since $\text{Vir}(e)$ is rational, $V$ is a semisimple $\text{Vir}(e)$-module. For $h = 0, 1/2, 1/16$, denote by $V_e(h)$ the sum of all irreducible $\text{Vir}(e)$-submodules of $V$ isomorphic to $L(\frac{1}{2}, h)$. Then we have the isotypical decomposition:

$$V = V_e(0) \oplus V_e(\frac{1}{2}) \oplus V_e(\frac{1}{16})$$

Define a linear automorphism $\tau_e$ on $V$ by

$$\tau_e = \begin{cases} 
1 & \text{on } V_e(0) \oplus V_e(\frac{1}{2}) \\
-1 & \text{on } V_e(\frac{1}{16}).
\end{cases}$$

Then, $\tau_e$ is an automorphism of $V$ with $\tau_e^2 = 1$. On the $\langle \tau_e \rangle$-fixed point subalgebra $V^{\langle \tau_e \rangle} = V_e(0) \oplus V_e(\frac{1}{2})$, define a linear automorphism $\sigma_e$ by

$$\sigma_e = \begin{cases} 
1 & \text{on } V_e(0) \\
-1 & \text{on } V_e(\frac{1}{2}).
\end{cases}$$

Then, $\sigma_e$ is an automorphism of $V^{\langle \tau_e \rangle}$ with $\sigma_e^2 = 1$. We will refer $\tau_e \in \text{Aut}(V)$ (resp. $\sigma_e \in \text{Aut}(V^{\langle \sigma_e \rangle})$) to as the $\tau$-involution (resp. $\sigma$-involution). An Ising vector $e$ of $V$ is called of $\sigma$-type if $\tau_e$ defines identity on $V$ i.e. $V_e(\frac{1}{16}) = 0$. 


We consider a VOA $V = \oplus_{n=0}^{\infty} V_n$ with $V_0 = \mathbb{C}1$ and $V_1 = 0$. Then the weight two subspace $V_2$ equipped with the product
\[ a \cdot b := a_{(1)}b, \ a, b \in V_2 \]
forms a commutative algebra with an symmetric bilinear form $\langle \cdot , \cdot \rangle$ defined by
\[ a_{(3)}b = \langle a, b \rangle 1, \ a, b \in V_2, \]
and satisfying
\[ \langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle, \ a, b, c \in V_2. \]
This algebra is called the Griess algebra of $V$. If $e \in V_2$ is a conformal vector with central charge $c$, $\frac{1}{2}e$ is an idempotent of the Griess algebra $V_2$ and $\langle e, e \rangle = \frac{c}{2}$.

About $\sigma$-involutions, the following is known.

**Theorem 2.1 (Miyamoto).** Assume that $V_0 = \mathbb{C}1$, $V_1 = 0$ and $\langle \cdot , \cdot \rangle$ is positive-definite. If $e, f \in V_2$ are Ising vectors of $\sigma$-type and $e \neq f$, then the order of $\sigma_e \sigma_f$ is 2 or 3, and
(1) If $|\sigma_e \sigma_f| = 2$, then $\langle e, f \rangle = 0$ and $e \cdot f = 0$.
(2) If $|\sigma_e \sigma_f| = 3$, then $\langle e, f \rangle = \frac{1}{32}$ and $e \cdot f = \frac{1}{4}(e + f - e^{\sigma_f})$.

3 Ising vectors of $V^+_{\sqrt{2}R}$

Let $R$ be a root lattice with root system $\Phi(R)$. Let $\ell$ be the rank of $R$ and $h$ the Coxeter number of $R$. We denote by $\sqrt{2}R$ the lattice whose norm is twice of $R$'s. Let $V_{\sqrt{2}R}$ be a lattice VOA associated to the lattice $\sqrt{2}R$. For any isometry $g$ on $R$, $g$ is extended to a linear automorphism of $V_{\sqrt{2}R}$ by setting
\[ \tilde{g}(\alpha_{(-n_1)}^{1} \ldots \alpha_{(-n_k)}^{k} e^{\sqrt{2}\alpha}) = g(\alpha_{(-n_1)}^{1}) \ldots g(\alpha_{(-n_k)}^{k}) e^{\sqrt{2}g(\alpha)} \]
for $\alpha^{1}, \ldots, \alpha^{k}, \alpha \in R$. This extension gives an automorphism of the VOA $V_{\sqrt{2}R}$ and $\tilde{g}$ is called a lift of $g$. We consider the lift $\theta$ of $(-1)$-isometry on $R$ and the fixed point subalgebra
\[ V^+_{\sqrt{2}R} = \{ v \in V | \theta(v) = v \} \]
of the lattice VOA $V_{\sqrt{2}R}$. It is clear that $V_{\sqrt{2}R}^+$ has a grading $V_{\sqrt{2}R}^+ = \oplus_{n \geq 0}(V_{\sqrt{2}R}^+)_n$ such that $(V_{\sqrt{2}R}^+)_0 = \mathbb{C}1$ and $(V_{\sqrt{2}R}^+)_1 = 0$, and

$$\omega = \frac{1}{4h} \sum_{\alpha \in \Phi(R)} \alpha(-1)^2 1$$

is the Virasoro vector of $V_{\sqrt{2}R}^+$. We give a classification of Ising vectors of $V_{\sqrt{2}R}^+$. For $\alpha \in \Phi(R)$ we set

$$\omega^\pm(\alpha) = \frac{1}{8} \alpha(-1)^2 1 \pm \frac{1}{4} \left(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}\right).$$

It is easy to show that $\omega^\pm(\alpha), \alpha \in \Phi(R)$, are Ising vectors of $\sigma$-type. of $V_{\sqrt{2}R}^+$. Set

$$s_R = \frac{2}{h+2} \sum_{\alpha \in \Phi(R)} \omega^-(\alpha)$$

$$= \frac{1}{4(h+2)} \sum_{\alpha \in \Phi(R)} \alpha(-1)^2 1 - \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha}$$

and

$$\tilde{\omega}_R = \omega - s_R$$

$$= \frac{2}{h+2} \omega + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha}.$$ 

Then $s_R$ and $\tilde{\omega}_R$ are mutually orthogonal Ising vectors which are fixed under the action of $\text{Aut}(R)$. The central charge $\tilde{c}_R$ of $\tilde{\omega}_R$ is given by the following:

<table>
<thead>
<tr>
<th>$R$</th>
<th>$A_n$</th>
<th>$D_n$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{c}_R$</td>
<td>$2n/(n+2)$</td>
<td>$1$</td>
<td>$6/7$</td>
<td>$7/10$</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>

In particular, $\tilde{\omega}_{E_8}$ is also an Ising vector of $\sigma$-type of $V_{\sqrt{2}R}^+$. For $x \in R$, define

$$\varphi_x = \exp\left(\frac{\pi \sqrt{-2}}{2} x(0)\right).$$

Then $\varphi_x$ is an automorphism of $V_{\sqrt{2}R}^+$ with $\varphi_2x = 1$. We set

$$I_R = \{\omega^\pm(\alpha) | \alpha \in \Phi(R)\},$$

$$\bar{I}_R = \{\varphi_x\tilde{\omega}_R | x \in R\}.$$

The inner products of these elements is given by

\[
\begin{align*}
\langle \omega^+(\alpha), \omega^-(\alpha) \rangle &= 0, \\
\langle \omega^\pm(\alpha), \omega^\mp(\beta) \rangle &= \langle \omega^\pm(\alpha), \omega^\mp(\beta) \rangle = \frac{1}{32}(\alpha, \beta)^2, \\
\langle \omega^+(\alpha), \varphi_x \tilde{\omega}_R \rangle &= \frac{1 \pm (-1)^{\langle x, \alpha \rangle}}{2(h+2)}, \\
\langle \tilde{\omega}_R, \varphi_x \tilde{\omega}_R \rangle &= \begin{cases} 
0 & \text{if } \langle x, x \rangle = 4 \\
\frac{1}{32} & \text{if } \langle x, x \rangle = 2 \\
\frac{1}{4} & \text{if } x \in 2E_8
\end{cases}
\end{align*}
\]

for distinct \( \alpha, \beta \in \Phi(R) \) and \( x \in R \).

It is known that \( V^{+}_{\sqrt{2}D_{2n}} \) and \( V^{+}_{\sqrt{2}E_{8}} \) are code VOAs and Lam classified Ising vectors of \( \sigma \)-type of a code VOA. We denote by \( I(V) \) the set of Ising vectors of a VOA \( V \). Then, the following hold.

**Theorem 3.1.** We have

1. \( I(V^{+}_{\sqrt{2}D_{2n}}) = I_{D_{2n}} \)
2. \( I(V^{+}_{\sqrt{2}E_{8}}) = I_{E_{8}} \cup \tilde{I}_{E_{8}} \)

Since a root lattice of ADE type is contained in \( E_8 \) or \( D_{2n} \) for sufficient large \( n \), by using the above theorem, the Ising vectors of \( V^{+}_{\sqrt{2}R} \) are given by the following.

**Theorem 3.2.** For any root lattice \( R \), \( I(V^{+}_{\sqrt{2}R}) = I_R \cup \left( \bigcup_{K \subset R, K \simeq E_8} \tilde{I}_K \right) \)

### 4 Commutant subalgebras \( M_R \)

For a VOA \( V \) and a conformal vector \( e \) of \( V \), we define the commutant subalgebra \( \text{Com}_V(e) \) by

\[ \text{Com}_V(e) = \{ v \in V | e_{(0)} v = 0 \} . \]

Let \( R \) be a root lattice and let us fix \( \gamma \in \Phi(E_8) \). We set

\[ M_R = \text{Com}_{V^{+}_{\sqrt{2}R}}(\tilde{\omega}_R) \]

and

\[ M'_{E_8} = \text{Com}_{V^{+}_{\sqrt{2}E_8}}(\tilde{\omega}_{E_8}) \cap \text{Com}_{V^{+}_{\sqrt{2}E_8}}(\omega^+(\gamma)). \]
We have $M_R \cap E = \{ e \in E \mid \langle \tilde{\omega}_R, e \rangle = 0 \}$ for a set $E$ of Ising vectors. By Theorem 3.2 and (*), the Ising vectors of $V_{\sqrt{2}R}^+$ are given by the following.

**Theorem 4.1.** (1) $I(M_R) = M_R \cap I(V_{\sqrt{2}R})$ and 
\[
M_R \cap I_R = \{ \omega^-(\alpha) \mid \alpha \in \Phi(R) \}, \\
M_E \cap I_E = \{ \varphi_x(\tilde{\omega}_E) \mid x \in E, \langle x, x \rangle = 4 \}.
\]
(2) $I(M_{E_8}'_R) = (M_{E_8}' \cap I_{E_8}) \cup (M_{E_8}' \cap I_{E_8})$ and 
\[
M_{E_8}' \cap I_{E_8} = \{ \omega^-(\alpha) \mid \alpha \in \Phi(E_8), \langle \alpha, \gamma \rangle \in 2\mathbb{Z} \}, \\
M_{E_8}' \cap I_{E_8} = \{ \varphi_x\tilde{\omega}_{E_8} \mid x \in E_8, \langle x, x \rangle = 4, \langle x, \gamma \rangle \in 1 + 2\mathbb{Z} \}.
\]

For $E \subset I(V)$ satisfying $\sigma_e(f) \in E$ for any $e, f \in E$, we define 
\[
\text{Aut}(E, \langle \cdot, \cdot \rangle) = \{ g \in \text{Sym}_E \mid \langle g(e), g(f) \rangle = \langle e, f \rangle, e, f \in E \}.
\]

Set 
\[
I_R^- = \{ \omega^-(\alpha) \mid \alpha \in \Phi(R) \}.
\]

Then the following hold.

**Proposition 4.2.** The map $\phi : \text{Aut}(R) \rightarrow \text{Aut}(I_R^-, \langle \cdot, \cdot \rangle), \ g \mapsto g|_{I_R^-}$ is a surjective group homomorphism with $\ker\phi = \langle -1 \rangle$. Therefore, 
\[
\text{Aut}(I_R^-, \langle \cdot, \cdot \rangle) \simeq \text{Aut}(R)/\langle -1 \rangle.
\]

On the other hand, we proved

**Theorem 4.3.** If $R$ is a root lattice of ADE type and VOA $V$ is $M_R$ or $M_{E_8}'$, 
(1) $V$ is generated by the weight 2 subspace $V_2$, in particular, by $I(V)$.
(2) The map $\text{Aut}(V) \rightarrow \text{Aut}(I(V), \langle \cdot, \cdot \rangle), \rho \mapsto \rho|_{I(V)}$ is an injective homomorphism.

By Proposition 4.2 and Theorem 4.3,

**Theorem 4.4.** If $R \neq E_8$, then $\text{Aut}(M_R) \simeq \text{Aut}(R)/\langle -1 \rangle$.

In the case that $R = E_8$, the following hold.

**Theorem 4.5.**
\[
\text{Aut}(M_{E_8}) \simeq \text{Aut}(I(M_{E_8}), \langle \cdot, \cdot \rangle) \simeq \text{Sp}_8(2) \\
\text{Aut}(M_{E_8}') \simeq \text{Aut}(I(M_{E_8}'), \langle \cdot, \cdot \rangle) \simeq \text{O}_8^-(2)
\]