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<thead>
<tr>
<th>Title</th>
<th>On Some Doubly Infinite, Finite and Mixed Sums derived from The N-fractional Calculus of A Power Function (Sakaguchi Functions in Univalent Function Theory and Its Applications)</th>
</tr>
</thead>
<tbody>
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</tr>
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On Some Doubly Infinite, Finite and Mixed Sums derived from The N-Fractional Calculus of A Power Function

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Abstract

In a previous paper, some doubly infinite, finite and mixed sums are reported using the N-fractional calculus \( ((z-c)^{a\cdot\beta})_{\gamma} \) by the author and his colleagues.

In this article the same doubly infinite sums in a previous paper are discussed again using \( ((z-c)^{\beta} \cdot (z-c)^{a})_{\gamma} \), the N-fractional calculus of products of power functions.

§ 0. Introduction (Definition of Fractional Calculus)

(1) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let \( D = \{D_{-}, D_{+}\} \), \( C = \{C_{-}, C_{+}\} \),

\( C_{-} \) be a curve along the cut joining two points \( z \) and \( -\infty + i \text{Im}(z) \),

\( C_{+} \) be a curve along the cut joining two points \( z \) and \( \infty + i \text{Im}(z) \),

\( D_{-} \) be a domain surrounded by \( C_{-} \), \( D_{+} \) be a domain surrounded by \( C_{+} \).

(Here \( D \) contains the points over the curve \( C \).)

Moreover, let \( f = f(z) \) be a regular function in \( D(z \in \mathbb{D}) \),

\[
f_{\nu}(z) = (f)_{\nu} = \frac{\Gamma(\nu+1)}{2\pi i} \int_{C} \frac{f(\zeta)}{(\zeta-z)^{\nu+1}} d\zeta \quad (\nu \notin \mathbb{Z}^{*}),
\]

\[
(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_{\nu} \quad (m \in \mathbb{Z}^{*}),
\]

where 

\[-\pi \leq \arg(\zeta-z) \leq \pi \quad \text{for} \quad C_{-}, \quad 0 \leq \arg(\zeta-z) \leq 2\pi \quad \text{for} \quad C_{+},
\]

\( \zeta \neq z, \quad z \in C, \quad \nu \in \mathbb{R}, \quad \Gamma; \text{Gamma function}, \)

then \( (f)_{\nu} \) is the fractional differintegration of arbitrary order \( \nu \) (derivatives of order \( \nu \) for \( \nu > 0 \), and integrals of order \( -\nu \) for \( \nu < 0 \)), with respect to \( z \), of the function \( f \), if \( |(f)_{\nu}| < \infty \).

(II) On the fractional calculus operator \( N^{\nu} [3] \)
Theorem A. Let fractional calculus operator (Nishimoto's Operator) $N^\nu$ be

$$N^\nu = \left( \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta-z)^{\nu+1}} \right) \quad (\nu \not\in \mathbb{Z}), \quad \text{[Refer to (1)]} \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in \mathbb{Z}^*), \tag{4}$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in \mathbb{R}\} \tag{6}$$

is an Abelian product group (having continuous index $\nu$) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator $N^\nu$, for the function $f$ such that $f \in F = \{f ; 0 < |f(z)| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in \mathbb{C}$. (For our convenience, we call $N^\beta \circ N^\alpha$ as product of $N^\beta$ and $N^\alpha$.)

Theorem B. "F.O.G. \{N^\nu\}" is an "Action product group which has continuous index $\nu$" for the set of $F$. (F.O.G.; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \tag{7}$$

Then the set $S$ is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(1) \((z-c)^\beta) = e^{-i\pi \beta} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} (z-c)^{\beta - \alpha} \quad \left( \left| \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} \right| < \infty \right), \tag{1}

(ii) \((\log(z-c)) = -e^{-i\pi \alpha} \Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty), \tag{i1} \)

(iii) \(((z-c)^{-\alpha}) = -e^{i\pi \alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty), \tag{i11} \)

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii). ($\Gamma$; Gamma function),

(iv) \((u \cdot v) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} u_{\alpha-k} v_k \quad \left( u = u(z), \ v = v(z) \right). \tag{iv} \)
§ 1. Doubly Infinite, Finite and Mixed Infinite Sums

In the following \(\alpha, \beta, \gamma \in \mathbb{R}\).

Theorem 1. Let

\[
L(\alpha, \beta, \gamma ; k, m) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(k-\alpha+m)\Gamma(\gamma-\beta-m)}{k!\cdot m!\Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(k-\alpha)\Gamma(-\beta)}.
\]

(1)

(i) When \(\alpha, \beta, \gamma \not\in \mathbb{Z}^*_0\), we have the following doubly infinite sums:

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma ; k, m) \left(\frac{-c}{z}\right)^k \left(\frac{z-c}{z}\right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z}\right)^{\alpha},
\]

where

\[
Q = Q(\alpha, \beta, \gamma) := \frac{\sin \pi \beta \cdot \sin \pi (\gamma-\alpha-\beta)}{\sin \pi (\alpha+\beta) \cdot \sin \pi (\gamma-\beta)} \quad (|Q|=M<\infty),
\]

and

\[
|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}|, \quad \frac{\Gamma(\gamma-\beta-m)}{\Gamma(-\beta)} < \infty.
\]

(ii) When \(\alpha, \beta \not\in \mathbb{Z}^*\), we have the following mixed infinite sums:

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, s ; k, m) \left(\frac{-c}{z}\right)^k \left(\frac{z-c}{z}\right)^m = Q(\alpha, \beta, s) \frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z}\right)^{\alpha},
\]

for \(s \in \mathbb{Z}^*\) where

\[
|\frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)}|, \quad \frac{\Gamma(s-m-\beta)}{\Gamma(-\beta)} < \infty.
\]

Proof of (i). We have

\[
(z-c)^\alpha = z^\alpha \left(1-\frac{c}{z}\right)^\alpha
\]

\[
= z^\alpha \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(a+1)}{k! \Gamma(a+1-k)} z^{-k} \quad (|z|>|c|)
\]

\[
= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(a+1)}{k! \Gamma(a+1-k)} z^{a-k}.
\]
Next make \((z - c)^\beta \times (7)\), then operate \(N\) to its both sides, we obtain

\[
((z - c)^\beta \cdot (z - c)^\alpha )_\gamma = \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} ((z - c)^\beta \cdot (z - c)^\alpha )_k
\]

\[
= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma + 1)}{m! \Gamma(\gamma + 1 - m)} ((z - c)^\beta )_\gamma(m, (z - c)^\alpha )_m.
\]

Now we have

\[
(z - c)^\beta )_\gamma = e^{-\pi (\gamma - m)} \frac{\Gamma(\gamma - m - \beta)}{\Gamma(-\beta)} (z - c)^{\beta - \gamma + m},
\]

and

\[
(z^a)_m = e^{-\pi m} \frac{\Gamma(m + k - \alpha)}{\Gamma(k - \alpha)} z^{\alpha - k}.
\]

respectively.

On the other hand we have

\[
((z - c)^\beta \cdot (z - c)^\alpha )_\gamma = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + 1)}{k! \Gamma(\gamma + 1 - k)} ((z - c)^\beta )_\gamma(k, (z - c)^\alpha )_k
\]

\[
= e^{-\pi (\gamma - k)} \frac{\Gamma(\gamma - k - \beta)}{\Gamma(-\beta)} (z - c)^{k - \gamma + \beta}.
\]

since

\[
(z - c)^\beta )_\gamma = e^{-\pi (\gamma - k)} \frac{\Gamma(\gamma - k - \beta)}{\Gamma(-\beta)} (z - c)^{\beta - \gamma + k},
\]

and

\[
((z - c)^\alpha )_k = e^{-\pi k} \frac{\Gamma(k - \alpha)}{\Gamma(-\alpha)} (z - c)^{\alpha - k},
\]

and

\[
\Gamma(\lambda + 1 - k) = (-1)^{-k} \frac{\Gamma(\lambda + 1)\Gamma(-\lambda)}{\Gamma(k - \lambda)}
\]
where
\[ [\lambda]_k = \lambda(\lambda+1) \cdots (\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda), \text{ with } [\lambda]_0 = 1 \]
(notation of Pochhammer).

Next we have the identity
\[
\sum_{k \geq 0} \frac{[a]_k [b]_k}{k! [c]_k} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \left\{ \begin{array}{l}
\text{Re}(c-a-b) > 0, \\
\text{Re}(c-a-b) \notin \mathbb{Z}_0
\end{array} \right. \] (18)

Therefore, we have
\[
((z-c)^{\beta} \cdot (z-c)^{\alpha})_\gamma = e^{-i\pi \gamma} \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)} (z-c)^{\alpha+\beta-\gamma} \, \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} \frac{\Gamma(\gamma+1) \Gamma(\gamma-m-\beta) \Gamma(m+k-\alpha)}{m! \Gamma(\gamma+1-m) \Gamma(-\beta) \Gamma(k-\alpha)}\] (19)

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+1) \Gamma(\gamma-m-\beta) \Gamma(m+k-\alpha)}{m! \Gamma(\gamma+1-m) \Gamma(-\beta) \Gamma(k-\alpha)} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} \frac{\Gamma(\gamma+1) \Gamma(\gamma-m-\beta) \Gamma(m+k-\alpha)}{m! \Gamma(\gamma+1-m) \Gamma(-\beta) \Gamma(k-\alpha)} \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)} (z-c)^{\alpha+\beta-\gamma} \]

we have then
\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma; k, m) \left( \frac{-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \frac{\Gamma(\gamma+1) \Gamma(\gamma-m-\beta) \Gamma(m+k-\alpha)}{m! \Gamma(\gamma+1-m) \Gamma(-\beta) \Gamma(k-\alpha)} \frac{\Gamma(\gamma+1) \Gamma(\gamma-m-\beta) \Gamma(m+k-\alpha)}{m! \Gamma(\gamma+1-m) \Gamma(-\beta) \Gamma(k-\alpha)} (z-c)^{\alpha+\beta-\gamma}, \quad (2)
\]
from (25), using the notation (1), under the conditions.

Proof of (ii). Set \( \gamma = s \in \mathbb{Z}^+ \) in (2), we have then (4) clearly under the conditions.

Corollary 1. When \( r, s \in \mathbb{Z}^+ \) we have the following doubly finite sums:

\[
\sum_{k=0}^{r} \sum_{m=0}^{s} L(r, \beta, s; k, m) \left( \frac{-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = Q(r, \beta, s) \frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)} \left( \frac{z-c}{z} \right)^r, \tag{26}
\]

where

\[ |c/z|, \quad |(z-c)/z| < \infty, \]

and

\[ \left| \frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)} \right|, \quad \left| \frac{\Gamma(s-\beta-m)}{\Gamma(-\beta)} \right| < \infty. \]

Proof. Set \( \alpha = r \) and \( \gamma = s \) in (2) we have then this corollary clearly.

§ 2. Direct calculation of the doubly infinite sums

The direct calculation (without the use of N-fractional calculus) of the doubly infinite sum in the LHS of §1 (2) is shown as follows.

Theorem 2. Let

\[
L = L(\alpha, \beta, \gamma; k, m) := \frac{\Gamma(\alpha+1) \Gamma(\gamma+1) \Gamma(\gamma-\beta-m) \Gamma(k-\alpha+m)}{k! \cdot m! \Gamma(\alpha+1-k) \Gamma(\gamma+1-m) \Gamma(-\beta) \Gamma(k-\alpha)}, \tag{1}
\]

and

\[
Q = Q(\alpha, \beta, \gamma) := \frac{\sin \pi \beta \cdot \sin \pi (\gamma-\alpha-\beta)}{\sin \pi (\alpha+\beta) \cdot \sin \pi (\gamma-\beta)} \quad (|Q(\alpha, \beta, \gamma)| = M < \infty). \tag{2}
\]

We have then

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \left( \frac{z-c}{z} \right)^m \left( \frac{-c}{z} \right)^k = Q \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left( \frac{z-c}{z} \right)^{\alpha}, \tag{3}
\]

where

\[ |c/z| < 1, \]

and

\[ (\alpha + \beta), \ (\gamma - \beta), \ (\gamma - \alpha - \beta) \notin \mathbb{Z}. \]

Proof. Now we have

\[
L \left( \frac{z-c}{z} \right)^m \left( \frac{-c}{z} \right)^k = \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)} \frac{\left[ -\alpha \right]_m [1-\gamma]_m}{k! \cdot m! [1+\beta-\gamma]_m} \left( \frac{c}{z} \right)^k \left( \frac{z-c}{z} \right)^m \tag{4}
\]
using the identity

\[ \Gamma(\lambda + 1 - k) = (-1)^{-k} \frac{\Gamma(\lambda + 1)\Gamma(-\lambda)}{\Gamma(k - \lambda)} \]  

(5)

and

\[ [\alpha]_{k+m} = [-\alpha]_m [-\alpha + m]_k. \]  

(6)

We have then

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \left( \frac{-c}{z} \right)^k \left( \frac{z-c}{z} \right)^m = \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \] 

\times \sum_{m=0}^{\infty} \frac{[-\alpha]_m [-\gamma]_m (z-c)^m}{m! (1 + \beta - \gamma)_m} \sum_{k=0}^{\infty} \frac{[-\alpha + m]_k (c/z)^k}{k!} \]  

(7)

\[
= \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \left( \frac{z-c}{z} \right)^\alpha \sum_{m=0}^{\infty} \frac{[-\alpha]_m [-\gamma]_m}{m! (1 + \beta - \gamma)_m} \]  

(8)

\[
= \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \left( \frac{z-c}{z} \right)^\alpha \; {}_2F_1(-\alpha, -\gamma; 1 + \beta - \gamma; z) \]  

(9)

\[
= \frac{\Gamma(\gamma - \beta)\Gamma(1 + \beta - \gamma)}{\Gamma(1 + \beta - \gamma)} \Gamma(1 + \alpha + \beta) \left( \frac{z-c}{z} \right)^\alpha, \]  

(10)

where

\[ \left| \frac{-c}{z} \right| < 1, \; \left| \frac{z-c}{z} \right| < 1, \; \text{Re}(\alpha + \beta) > -1. \]  

Because we ahve

\[
\sum_{k=0}^{\infty} \frac{[-\alpha + m]_k (c/z)^k}{k!} = \left( \frac{z-c}{z} \right)^{\alpha - m} \]  

(11)

since

\[
\sum_{k=0}^{\infty} \frac{[\lambda]_k z^k}{k!} = (1 - z)^{-\lambda}, \]  

(12)

and

\[
{}_2F_1(a, b; c; 1) = \sum_{m=0}^{\infty} \frac{[a]_m [b]_m}{m! [c]_m} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left( \frac{\text{Re}(c-a-b) > 0}{c \notin \mathbb{Z}_0} \right). \]  

(13)

Moreover we have the identity

\[
\Gamma(\lambda)\Gamma(1 - \lambda) = \frac{\pi}{\sin \pi \lambda} \quad (\lambda \notin \mathbb{Z}), \]  

(14)

then applying (14) to (10) we obtain

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \left( \frac{z-c}{z} \right)^m \left( \frac{-c}{z} \right)^k = Q \frac{\Gamma(\gamma - \alpha - \beta) (z-c)^\alpha}{\Gamma(-\alpha - \beta) \left( \frac{z-c}{z} \right)^\alpha}. \]  

(15)
§ 3. Commentary

[1] In a previous paper, the results obtained by the author are derived by the use of \((z-c)^{\alpha + \beta}\), however the results shown in this article, the N-fractional calculus \(((z-c)^{\alpha} \cdot (z-c)^{\beta}\rangle\) is used.

[11] When \(Q = Q(\alpha, \beta, \gamma) = 1\), § 1. (2) overlaps Theorem 2 obtained in a previous paper. [11]

References


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