SEVERAL REVERSE INEQUALITIES OF OPERATORS

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ABSTRACT. In this report, we show reverse inequalities to Araki’s inequality and investigate the equivalence among reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities. Among others, we show that if \( A \) and \( B \) are positive operators on a Hilbert space \( H \) such that \( 0 < mI \leq A \leq MI \) for some scalars \( m < M \), then
\[
K(m, M, p) \| BAB \|^{p} \leq \| B^{p}A^{p}B^{p} \| \quad \text{for all } 0 < p < 1,
\]
where \( K(m, M, p) \) is a generalized Kantorovich constant by Furuta.

1. INTRODUCTION

Let \( A \) and \( B \) be positive operators on a Hilbert space \( H \). The equivalence among Cordes and Löwner-Heinz inequalities was discussed by many authors. In [8], Furuta showed that the Cordes inequality
\[
\| A^{p}B^{p} \| \leq \| AB \|^{p} \quad \text{for } 0 < p < 1
\]
is equivalent to the Löwner-Heinz inequality (cf.[14])
\[
A \geq B \geq 0 \quad \text{implies } \quad A^{p} \geq B^{p} \quad \text{for } 0 < p < 1
\]
(cf. [5]). In [1], Araki showed a trace inequality which entails the following inequality:
\[
\| B^{p}A^{p}B^{p} \| \leq \| BAB \|^{p} \quad \text{for } 0 < p < 1.
\]
Moreover, it was shown in [6, 2] that the Cordes inequality (1) is equivalent to Araki’s inequality (3).

On the other hand, Furuta [9] showed the following Kantorovich type inequalities: If \( A \) and \( B \) are positive operators with \( 0 < mI \leq A \leq MI \) for some scalars \( m < M \), then
\[
A \geq B \geq 0 \quad \text{implies } \quad K(m, M, p)A^{p} \geq B^{p} \quad \text{for } p > 1,
\]
where a generalized Kantorovich constant \( K(m, M, p) \) [3, 7, 11] is defined as
\[
K(m, M, p) = \frac{mM^{p} - Mm^{p}}{(p-1)(M-m)} \left( \frac{p-1}{p} \frac{M^{p} - m^{p}}{mM^{p} - Mm^{p}} \right)^{p}
\]
for all real numbers \( p \).

We here cite Furuta’s textbook [10] as a pertinent reference to Kantorovich inequalities.

Also, Yamazaki [16] showed the following difference type reverse inequalities of the Löwner-Heinz inequality: If \( A \) and \( B \) are positive operators with \( 0 < mI \leq B \leq MI \) for some scalars \( m < M \), then
\[
A \geq B \geq 0 \quad \text{implies } \quad C(m, M, p) + A^{p} \geq B^{p} \quad \text{for } p > 1,
\]
where the constant $C(m, M, p)$ [12, 16] is defined as

$$C(m, M, p) = (p - 1) \left( \frac{M^p - m^p}{p(M - m)} \right)^{\frac{1}{p-1}} + \frac{Mm^p - mM^p}{M - m}$$

for all real numbers $p$.

In this report, we show reverse inequalities to Araki's inequality (3) and the Cordes inequality (1): If $A$ and $B$ are positive operators with $0 < mI \leq A \leq MI$ for some scalars $m < M$, then the following inequalities hold

$$K(m, M, p) \|BAB\|^p \leq \|B^pA^pB^p\|$$

for $0 < p < 1$, (8)

$$K(m^2, M^2, p)^{1/2} \|AB\|^p \leq \|A^pB^p\|$$

for $0 < p < 1$, (9)

respectively. We moreover show that reverse inequalities (4), (8) and (9) are mutually equivalent.

2. Main Results

First of all, we present our main theorem which is a reverse inequality to Araki's inequality (3).

**Theorem 1.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then for each $\alpha > 0$

$$\|BAB\|^p \leq \beta(m, M, p, \alpha)\|B^pA^pB^p\|^2$$

for all $0 < p < 1$, (10)

or equivalently

$$\|B^pA^pB^p\|^{\frac{1}{p}} \leq \alpha \|BAB\| + \beta(m, M, p, \alpha)\|B\|^2$$

for all $p > 1$, (11)

where

$$\beta(m, M, p, \alpha) = \begin{cases} \frac{1}{p} \left( \frac{M^p - m^p}{pm^p - m^p} \right) \frac{1}{p-1} & \text{if } \frac{M^p - m^p}{pM^{p-1}(M - m)} \leq \alpha \leq \frac{M^p - m^p}{pm^{p-1}(M - m)}, \\ (1 - \alpha)M & \text{if } 0 < \alpha \leq \frac{M^p - m^p}{pM^{p-1}(M - m)}, \\ (1 - \alpha)m & \text{if } \alpha \geq \frac{M^p - m^p}{pm^{p-1}(M - m)}. \end{cases}$$

If we choose $\alpha$ satisfying $\beta(m, M, p, \alpha) = 0$ in Theorem 1, then we have the following ratio type reverse inequalities.

**Corollary 2.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$K(m, M, p)\|BAB\|^p \leq \|B^pA^pB^p\|$$

for $0 < p < 1$, (13)

or equivalently

$$\|BAB\|^p \leq K(m, M, p)\|B^pA^pB^p\|$$

for $p > 1$, (14)

where $K(m, M, p)$ is defined as (5) in §1.
If we put \( \alpha = 1 \) in Theorem 1, then we have the following difference type reverse inequalities.

**Corollary 3.** If \( A \) and \( B \) are positive operators on \( H \) such that \( 0 < mI \leq A \leq MI \) for some scalars \( m < M \), then

\[
\|BAB\|^p - \|B^pA^pB^p\| \leq -C(m, M, p)\|B\|^{2p} \quad \text{for } 0 < p < 1,
\]

or equivalently

\[
\|B^pA^pB^p\|^{\frac{1}{p}} - \|BAB\| \leq -C(m^p, M^p, \frac{1}{p})\|B\|^2 \quad \text{for } p > 1,
\]

where \( C(m, M, p) \) is defined as (7) in §1.

As special cases of Corollary 2 and Corollary 3, we have the following corollary.

**Corollary 4.** If \( A \) and \( B \) are positive operators on \( H \) such that \( 0 < mI \leq A \leq MI \) for some scalars \( m < M \), then

\[
\|B^2A^2B^2\| \leq \frac{(M+m)^2}{4Mm}\|BAB\|^2.
\]

\[
\|B^2A^2B^2\|^{\frac{1}{2}} - \|BAB\| \leq \frac{(M-m)^2}{4(M+m)}\|B\|^2.
\]

\[
\frac{2\sqrt{Mm}}{\sqrt{M}+\sqrt{m}}\|BAB\|^{\frac{1}{2}} \leq \|B^\frac{1}{2}A^\frac{1}{2}B^\frac{1}{2}\|.
\]

\[
\|BAB\|^{\frac{1}{2}} - \|B^\frac{1}{2}A^\frac{1}{2}B^\frac{1}{2}\| \leq \frac{\sqrt{M} - \sqrt{m})^2}{4(\sqrt{M} + \sqrt{m})}\|B\|.
\]

Since \( \|X^*X\| = \|X\|^2 \) for an operator \( X \), we obtain the following reverse inequality to the Cordes inequality by Corollary 2.

**Theorem 5.** If \( A \) and \( B \) are positive operators on \( H \) such that \( 0 < mI \leq A \leq MI \) for some scalars \( m < M \), then

\[
K(m^2, M^2, p)^\frac{1}{2}\|AB\|^p \leq \|A^pB^p\| \quad \text{for all } 0 < p < 1,
\]

or equivalently

\[
\|A^pB^p\| \leq K(m^2, M^2, p)^\frac{1}{2}\|AB\|^p \quad \text{for all } p > 1.
\]

In particular,

\[
\sqrt{\frac{2\sqrt{Mm}}{M+m}\|AB\|^{\frac{1}{2}}} \leq \|A^\frac{1}{2}B^\frac{1}{2}\|.
\]

and

\[
\|A^2B^2\| \leq \frac{M^2 + m^2}{2Mm}\|AB\|^2
\]

The equivalence among the reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities is now given as follows.
Theorem 6. For a given $p > 1$, the following are mutually equivalent: For all $A, B \geq 0$ and $0 < mI \leq A \leq MI$

(A) $A \geq B \geq 0$ implies $K(m, M, p)A^p \geq B^p$.

(B) $\|AB\|^p \leq K(m^2, M^2, p)^{1/2} \|AB\|^p$.

(C) $\|B^pA^pB^p\| \leq K(m, M_{1/p}) \|BAB\|^p$.

(B') $K(m^2, M^2, 1/p)^{1/2} \|AB\|^p \leq \|AB\|^p$.

(C') $K(m, M, 1/p) \|BAB\|^p \leq \|B^pA^pB^p\|$.

3. Lemmas

We start with the following three lemmas before we give proofs of the results in §2.

Let $A$ be a positive operator on a Hilbert space $H$ and $x$ a unit vector in $H$. Then it follows from Hölder-McCarthy inequality that

\[(Ax, x) \leq (A^px, x)^{\frac{1}{p}} \text{ for all } p > 1.\]  

By using the Mond-Pečarić method [12, 13], we have the following reverse inequality of (25) [15, 4]:

Lemma 7. If $A$ is a positive operator on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$

\[(A^px, x)^{\frac{1}{p}} \leq \alpha(Ax, x) + \beta(m, M, p, \alpha) \quad \text{for all } p > 1 \]

holds for every unit vector $x \in H$, where $\beta(m, M, p, \alpha)$ is defined as (12) in Theorem 1.

Proof. For the sake of reader’s convenience, we give a proof. Put $\beta = \beta(m, M, p, \alpha)$ and $f(t) = (at+b)^{\frac{1}{p}} - \alpha t$ for $a = \frac{M^p-m^p}{M-m}$ and $b = \frac{Mm^{p-1}M^p-m^{p-1}m^p}{M-m}$, then we have $f'(t) = \frac{a}{p}(at+b)^{\frac{1}{p}-1} - \alpha$. It follows that the equation $f'(t) = 0$ has exactly one solution $t_0 = \frac{1}{a}(\frac{\alpha p}{a})^{\frac{p}{1-p}} - \frac{b}{a}$. If $m \leq t_0 \leq M$, then we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0)$ since $f''(t) = \frac{a^2(1-p)}{p^2 M^{p-1}(M-m)}(at+b)^{\frac{1}{p}-2} < 0$ and the condition $m \leq t_0 \leq M$ is equivalent to the condition

\[\frac{M^p-m^p}{pM^{p-1}(M-m)} \leq \alpha \leq \frac{M^p-m^p}{pm^{p-1}(M-m)}.\]

If $M \leq t_0$, then $f(t)$ is increasing on $[m, M]$ and hence we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1-\alpha)M$ for $t_0 = M$. Similarly, we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1-\alpha)m$ for $t_0 = m$ if $t_0 \leq m$. Hence it follows that

\[(at+b)^{\frac{1}{p}} - \alpha t \leq \beta \quad \text{for all } t \in [m, M].\]

Since $t^p$ is convex for $p > 1$, it follows that $t^p \leq at + b$ for $t \in [m, M]$. By the spectral theorem, we have $A^p \leq aA + b$ and hence $(A^px, x) \leq a(Ax, x) + b$ for every unit vector $x \in H$. Therefore we have

\[(A^px, x)^{\frac{1}{p}} - \alpha(Ax, x) \leq \alpha(Ax, x) + b)^{\frac{1}{p}} - \alpha(Ax, x) \leq \max_{m \leq t \leq M} f(t) = \beta(m, M, p, \alpha).\]
By Lemma 7, we have the following estimates of both the difference and the ratio in the inequality (25).

**Lemma 8.** If $A$ is a positive operator on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $p > 1$

\[(27) \quad (A^p x, x)^{\frac{1}{p}} \leq K(m, M, p)^{\frac{1}{p}} (Ax, x)\]

and

\[(28) \quad (A^p x, x)^{\frac{1}{p}} - (Ax, x) \leq C(m^{p}, M^{p}, \frac{1}{p})\]

hold for every unit vector $x \in H$, where $K(m, M, p)$ is defined as (5) in §1 and $C(m, M, p)$ is defined as (7) in §1.

**Proof.** If we choose $\alpha$ satisfying $\beta(m, M, p, \alpha) = 0$ in Lemma 7, then we have $\alpha = K(m, M, p)^{\frac{1}{p}}$. If we put $\alpha = 1$ in Lemma 7, then we have $\beta(m, M, p, 1) = -C(m^{p}, M^{p}, \frac{1}{p})$. \[\square\]

We remark that $K(m, M, 2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ if $p = 2$.

We summarize some important properties of a generalized Kantorovich constant [3, 11].

**Lemma 9.** Let $m < M$ be given. Then a generalized Kantorovich constant $K(m, M, p)$ has the following properties.

(i) $K(m, M, p) = K(M, m, p)$ for all $p \in \mathbb{R}$.

(ii) $K(m, M, p) = K(m, M, 1-p)$ for all $p \in \mathbb{R}$.

(iii) $K(m, M, 0) = K(m, M, 1) = 1$ for all $p \in \mathbb{R}$.

(iv) $K(m, M, p)$ is increasing for $p > \frac{1}{2}$ and decreasing for $p < \frac{1}{2}$.

(v) $K(m^{p}, M^{p}, \frac{p}{r})^{\frac{1}{p}} = K(m^{p}, M^{p}, \frac{r}{p})^{-\frac{1}{r}}$ for $pr \neq 0$.

4. **Proof of Results**

Based on Lemmas in the preceding section, we give proofs of the results mentioned in the second section.

**Proof of Theorem 1.**

For every unit vector $x \in H$, it follows that

\[\begin{align*}
((BAB)^p x, x) & \leq (BABx, x)^p \quad \text{by Hölder-McCarthy inequality and } 0 < p < 1 \\
& = \left( (A^p)^{\frac{1}{2}} B x \frac{B x}{\|B x\|} \right)^p \|B x\|^{2p} \\
& \leq \left( \alpha(A^p B x, B x) + \beta(m^{p}, M^{p}, \frac{1}{p}, \alpha) \right) \|B x\|^{2p} \quad \text{by Lemma 7} \\
& = \alpha(A^p B x, B x)\|B x\|^{2p-2} + \beta(m^{p}, M^{p}, \frac{1}{p}, \alpha)\|B x\|^{2p} \\
& = \alpha \left( B^p A^p B x \frac{B^p x}{\|B^p x\|}, \frac{B^p x}{\|B^p x\|} \right) \|B x\|^{2p-2}\|B^p x\|^2 + \beta(m^{p}, M^{p}, \frac{1}{p}, \alpha)\|B x\|^{2p}
\end{align*}\]
and
\[ \|Bx\|^{2p-2}\|B^{1-p}x\|^2 = (B^2x, x)^{p-1}(B^{2-2p}x, x) \leq (B^2x, x)^{p-1}(B^{2p-2}x, x)^{1-p} = 1 \quad \text{by } 0 < 1 - p < 1. \]

By combining two inequalities above, we have
\[
\|BAB\|^p = \|(BAB)^p\|
\leq \alpha\|B^pA^pB^p\| + \beta(m^p, M^p, \frac{1}{p}, \alpha)\|B\|^{2p}.
\]

and hence we have the desired inequality (10).

Next, we show (10)\(\Rightarrow\) (11). For \(p > 1\), since \(0 < \frac{1}{p} < 1\), it follows from (10) that
\[
\|BAB\|^\frac{1}{p} \leq \alpha \|B^\frac{1}{p}A^\frac{1}{p}B^\frac{1}{p}\| + \beta(m^\frac{1}{p}, M^\frac{1}{p}, p, \alpha)\|B\|^\frac{2}{p}.
\]

By replacing \(A\) by \(A^p\) and \(B\) by \(B^p\) in the above inequality respectively, we have
\[
\|B^pA^pB^p\|^\frac{1}{p} \leq \alpha \|BAB\| + \beta(m, M, p, \alpha)\|B^p\|^\frac{2}{p},
\]
and so we have the desired inequality (11). Similarly we can show (11)\(\Rightarrow\) (10). Therefore (10) is equivalent to (11).

\(\square\)

**Proof of Corollary 2.**
For \(p > 1\), if we put \(\beta(m, M, p, \alpha) = 0\) in Theorem 1, then it follows that
\[
p - 1 \left( \frac{M^p - m^p}{p(M - m)} \right)^{p-1} + \alpha^{\frac{p}{p-1}} \left( \frac{mM^p - m^p}{M^p - m^p} \right) = 0
\]
and hence
\[
\alpha^{\frac{p}{p-1}} = -\frac{p - 1}{p} \left( \frac{M^p - m^p}{p(M - m)} \right)^{p-1} \frac{M^p - m^p}{mM^p - mM^p}.
\]

Therefore, we have
\[
\alpha^p = \frac{M^p - m^p}{p(M - m)} \left( \frac{p - 1}{p} \frac{M^p - m^p}{mM^p - mM^p} \right)^{p-1}
= K(m, M, p)
\]
and we obtain the desired inequality (14).

For \(0 < p < 1\), since \(1/p > 1\), it follows from (14) that
\[
\|BAB\|^\frac{1}{p} \leq K(m, M, \frac{1}{p})\|B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}\|.
\]

By replacing \(A\) and \(B\) by \(A^p\) and \(B^p\) respectively, then we have
\[
\|B^pA^pB^p\|^\frac{1}{p} \leq K(m^p, M^p, \frac{1}{p})\|BAB\|.
\]

Hence it follows from Lemma 9 that
\[
\|B^pA^pB^p\| \leq K(m^p, M^p, \frac{1}{p})^p\|BAB\|^p
\leq K(m, M, p)^{-1}\|BAB\|^p,
\]
and we have the desired inequality (13). Similarly we have the implication (13)\(\Rightarrow\) (14). \(\square\)
Proof of Corollary 3.
If we put $\alpha = 1$ in Theorem 1, then it follows that
\[
\beta(m^p, M^p, \frac{1}{p}, 1) = \left( \frac{1}{\frac{1}{p}} \right) \left( \frac{p(M - m)}{M^p - m^p} \right) + \frac{M^p m - m^p M}{M - m} = (1 - p) \left( \frac{p(M - m)}{M^p - m^p} \right)^{\frac{p}{1-p}} + \frac{M^p m - m^p M}{M - m} = -C(m, M, p).
\]
Similarly it follows that $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. Hence we have the equivalence (15) $\iff$ (16) \hfill $\square$

Proof of Corollary 4.
In Corollary 2 and 3, we have only to put $p = 2$ and $p = 1/2$. \hfill $\square$

Proof of Theorem 5
By Corollary 2, it follows that
\[
K(m, M, p) \|A^\frac{1}{2}B\|^2p \leq \|A^\frac{p}{2}B^p\|^2.
\]
By replacing $A$ by $A^2$, we have
\[
K(m^2, M^2, p) \|AB\|^2p \leq \|A^pB^p\|^2.
\]
Therefore we have (21). Similarly, we have the equivalence (21) $\iff$ (22). \hfill $\square$

Proof of Theorem 6
The proof is divided into three parts, namely the equivalence $(A) \implies (B) \implies (C) \implies (A), (B) \iff (B')$ and $(C) \iff (C')$.

$(A) \implies (B)$. It follows that
\[
\|A^{-\frac{1}{2}}B^\frac{1}{2}\| \leq 1 \implies \|A^{-\frac{p}{2}}B^\frac{p}{2}\|^2 \leq K(m, M, p)
\]
\[
\iff \|A^\frac{1}{2}B^\frac{1}{2}\| \leq 1 \implies \|A^\frac{p}{2}B^p\|^2 \leq K(M^{-1}, m^{-1}, p) = K(m, M, p)
\]
\[
\iff \|AB\| \leq 1 \implies \|A^pB^p\| \leq K(m^2, M^2, p).
\]
If we put $B_1 = B/\|AB\|$, then it follows from $\|AB_1\| = 1$ that
\[
\|A^pB_1^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \iff \|A^pB^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \|AB\|^p.
\]

$(B) \implies (C)$. If we replace $A$ by $A^\frac{1}{2}$ in (A), then it follows that
\[
\|A^\frac{p}{2}B^\frac{p}{2}\| \leq K(m, M, p)^{\frac{1}{2}} \|A^\frac{1}{2}B^\frac{1}{2}\|^p.
\]
Square both sides, we have
\[
\|B^pA^pB^p\| \leq K(m, M, p) \|BAB\|^p.
\]

$(C) \implies (A)$. If we replace $B$ by $B^\frac{1}{2}$ and $A$ by $A^{-1}$ in (C), then it follows that
\[
\|B^\frac{p}{2}A^{-p}B^\frac{p}{2}\| \leq K(M^{-1}, m^{-1}, p) \|B^\frac{1}{2}A^{-1}B^\frac{1}{2}\|^p.
\]
By rearranging it, we have
\[
\|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \leq K(m, M, p) \|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\|^p.
\]
Since $A \geq B \geq 0$, it follows from $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq 1$ that
\[ \|A^{-\frac{p}{2}}B^{p}A^{-\frac{1}{2}}\| \leq K(m, M, p) \]
and hence
\[ B^{p} \leq K(m, M, p)A^{p}. \]

$(B) \iff (B')$: If we replace $A$ and $B$ by $A^{\frac{1}{p}}$ and $B^{\frac{1}{p}}$ in $(B)$ respectively, then it follows that
\[
(B) \iff \|AB\| \leq K(m^{\frac{2}{p}}, M^{\frac{2}{p}}, p)^{\frac{1}{2}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\|^{p} \\
\iff \|AB\|^{rac{1}{p}} \leq K(m^{\frac{2}{p}}, M^{\frac{2}{p}}, p)^{\frac{1}{2p}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \\
\iff (B')
\]
Similarly we have $(C) \iff (C')$ and so the proof is complete. \qed

REFERENCES