On the uniqueness of nodal radial solutions of sublinear elliptic equations in a ball

岡山理科大学・理学部 田中 敏 (Satoshi Tanaka)
Department of Applied Mathematics
Faculty of Science
Okayama University of Science

1. INTRODUCTION

We consider the second order ordinary differential equation

\[ u'' + \frac{N-1}{r}u' + K(r)f(u) = 0, \quad 0 < r < 1, \]

with the boundary condition

\[ u'(0) = u(1) = 0, \]

where \( N \geq 2, K \in C^2[0,1], K(r) > 0 \) for \( 0 \leq r \leq 1, f \in C^1(\mathbb{R}), sf(s) > 0 \) for \( s \neq 0 \). Assume moreover that the following sublinear condition is satisfied:

\[ \frac{f(s)}{s} > f'(s) \quad \text{for } s \neq 0. \]

Note that a solution of problem (1.1)-(1.2) is a radial solution \( u(r) \ (r = |x|) \) of the Dirichlet problem of

\[
\begin{cases}
\Delta u + K(|x|)f(u) = 0 & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
\]

where \( B = \{x \in \mathbb{R}^N : |x| < 1\} \).

We consider solutions \( u \) of problem (1.1)-(1.2) satisfying \( u(0) > 0 \) only. If \( u \) is a solution of problem (1.1)-(1.2) with \( u(0) < 0 \), then it can be treated similarly as in the case where \( u(0) > 0 \), since \( v \equiv -u \) satisfies \( v(0) > 0 \) and is a solution of

\[
\begin{cases}
v'' + \frac{N-1}{r}v' + K(r)f_0(v) = 0, & 0 < r < 1, \\
v'(0) = v(1) = 0,
\end{cases}
\]

where \( f_0(s) = -f(-s) \).

In this paper we study the uniqueness of solutions of the problem (1.1)-(1.2) having exactly \( k-1 \) zeros in \( (0,1) \), where \( k \in \mathbb{N} \).

Hence we consider the following problem:

\[
\begin{cases}
u'' + \frac{N-1}{r}u' + K(r)f(u) = 0, & 0 < r < 1, \\
u'(0) = u(1) = 0, & u(0) > 0,
\end{cases}
\]

(\( P_k \))

u has exactly \( k-1 \) zeros in \( (0,1) \).
It is known that there exists at least one solution of \((P_k)\) under a certain condition. For example, in the case where \(f(u) = |u|^{p-1}u, p > 0, p \neq 1\) and \(N \geq 3\), the existence results of solutions of \((P_k)\) were obtained by Y. Naito [4]. Assume that there exists limits \(f_0\) and \(f_\infty\) such that

\[
f_0 = \lim_{u \to 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u} \quad (0 \leq f_0, \ f_\infty \leq \infty).
\]

In the case where there is a sufficiently large gap between \(f_0\) and \(f_\infty\), the existence of solutions of \((P_k)\) was established by Dambrosio [1].

Now we consider the uniqueness of solutions of \((P_k)\). For the superlinear case \(f(u) = |u|^{p-1}u (p > 1)\), Yanagida [6] showed that, for each \(k \in \mathbb{N}\), \((P_k)\) has at most one solution if \(rK'(r)/K(r)\) is nonincreasing and \(N \geq 3\). For the sublinear case where \((3), f_0 = \infty\) and \(f_\infty = 0\), Kajikiya [2] proved that, for each \(k \in \mathbb{N}\), the solution of \((P_k)\) exists and is unique if \(K(r) \equiv 1\). However very little is known about the uniqueness of solutions of \((P_k)\) for the sublinear case and \(K(r) \not\equiv 1\).

The main result of this paper is as follows.

**Theorem 1.1.** Suppose that (1.3) holds. If

\[
3r^2(K')^2 - 2r^2 KK'' + 2(N-1)r KK' + 4(N-1)K^2 \geq 0, \quad 0 \leq r \leq 1,
\]

then, for each \(k \in \mathbb{N}\), \((P_k)\) has at most one solution.

In view of the following equality

\[
3r^2(K')^2 - 2r^2 KK'' + 2(N-1)r KK' + 4(N-1)K^2
\]

\[
= K^2 \left[ \left( \frac{rK'}{K} + 2 \right) \left( \frac{rK'}{K} + 2(N-1) \right) - 2r \left( \frac{rK'}{K} \right)' \right],
\]

we have the following corollary of Theorem 1.1.

**Corollary 1.1.** Suppose that (1.3) holds. Assume moreover that one of the following (1.5)–(1.7) is satisfied:

(1.5) \(K'' \leq 0, \ K' \geq 0 \) for \(0 \leq r \leq 1\),

(1.6) \(N = 2, \ \left( \frac{rK'}{K} \right)' \leq 0 \) for \(0 \leq r \leq 1\),

(1.7) \(N > 2, \ \frac{rK'}{K} \geq -2, \ \left( \frac{rK'}{K} \right)' \leq 0 \) for \(0 \leq r \leq 1\).

Then, for each \(k \in \mathbb{N}\), \((P_k)\) has at most one solution.

2. LEMMAS

In this section we give several lemmas.

First we note that (1.1) can be rewritten as follows:

\[
(r^{N-1}u')' + r^{N-1}K(r)f(u) = 0, \quad 0 < r < 1.
\]
The proof of Theorem 1.1 is based on the method of Kolodner [3]. Namely we consider the solution $u(r, \alpha)$ of (1.1) satisfying the initial condition

$$
(2.2) \quad u(0) = \alpha > 0, \quad u'(0) = 0,
$$

where $\alpha > 0$ is a parameter. Since $K \in C^2[0,1]$ and $f \in C^1(\mathbb{R})$, we see that $u(r, \alpha)$ exists on $[0,1]$ is unique and satisfies $u, u' \in C^1([0,1] \times (0, \infty))$, and that $u_\alpha(r, \alpha)$ is a solution of linearized problem

$$
(2.3) \quad \begin{cases}
(r^{N-1}u')' + r^{N-1}K(r)f'(u(r, \alpha))w = 0, & r \in (0,1], \\
w(0) = 1, & w'(0) = 0.
\end{cases}
$$

(See, for example, [5, §6 and 13].)

Hereafter we assume that $u(r, \alpha)$ is a solution of $(P_k)$. Let $z_i$ be the $i$-th zero of $u(r, \alpha)$. Let $t_1 = 0$. For each $i \in \{2, 3, \ldots, k\}$, there exists $t_i \in (z_{i-1}, z_i)$ such that $u'(t_i, \alpha) = 0$, since $u(r, \alpha)(r^{N-1}u'(r, \alpha))' < 0$ for $r \in (z_i, z_{i+1})$. Therefore we find that

$$
0 = t_1 < z_1 < t_2 < z_2 < \cdots < t_{k-1} < z_{k-1} < t_k < z_k = 1,
$$

$$
u(z_i, \alpha) = 0, \quad u'(t_i, \alpha) = 0, \quad i = 1, 2, \ldots, k,
$$

$$u(r, \alpha) > 0 \text{ for } r \in [t_1, z_1),
$$

$$(-1)^i u(r, \alpha) > 0 \text{ for } r \in (z_i, z_{i+1}), \quad i = 1, 2, \ldots, k-1,
$$

$$(-1)^i u'(r, \alpha) > 0 \text{ for } r \in (t_i, t_{i+1}), \quad i = 1, 2, \ldots, k-1,
$$

$$(-1)^k u'(r, \alpha) > 0 \text{ for } r \in (t_k, z_k].
$$

**Lemma 2.1.** Assume that (1.3) holds. Let $w$ be the solution of (2.3). Then $w(r) > 0$ for $x \in [0, z_1]$.

**Proof.** Note that $w(0) = 1$ and $w'(0) = 0$. Assume to the contrary that there exists a number $r_1 \in (0, z_1]$ such that $w(r) > 0$ for $r \in [0, r_1)$ and $w(r_1) = 0$. Then
we see that $w'(r_1) < 0$. Let $u \equiv u(r, \alpha)$. An easy computation shows that
\begin{equation}
[r^{N-1}(w'u - wu')]' = r^{N-1}K(r)[f(u) - f'(u)u]w.
\end{equation}
Recall that $u(r) > 0$ for $r \in [0, z_1)$. Integrating of (2.7) over $[0, r_1]$ and using (1.3), we have
\begin{align*}
 r_1^{N-1}w'(r_1)u(r_1) &= \int_0^{r_1} r^{N-1}K(r)[f(u) - f'(u)u]w \, dr > 0,
\end{align*}
which implies $w'(r_1) > 0$. This is a contradiction. Consequently we find that $w(r) > 0$ for $r \in (0, z_1]$.

**Lemma 2.2.** Assume that (1.3) holds. For each $i \in \{1, 2, \ldots, k-1\}$, the solution $w$ of (2.3) has at most one zero in $[z_i, z_{i+1}]$.

**Proof.** Note that $u \equiv u(r, \alpha)$ is a solution of
\begin{equation*}
(r^{N-1}u')' + r^{N-1}K(r)\frac{f(u)}{u}u = 0, \quad r \in (z_i, z_{i+1})
\end{equation*}
and satisfies $u(z_i) = u(z_{i+1}) = 0$ and $u(r) \neq 0$ for $r \in (z_i, z_{i+1})$. From (1.3) it follows that
\begin{equation*}
 r^{N-1}K(r)f'(u) < r^{N-1}K(r)\frac{f(u)}{u}, \quad r \in (z_i, z_{i+1}).
\end{equation*}
Assume to the contrary that there exist numbers $r_0$ and $r_1$ such that $z_i \leq r_0 < r_1 \leq z_{i+1}$ and $w(r_0) = w(r_1) = 0$. Then Sturm's comparison theorem implies that $u$ has at least one zero in $(r_0, r_1)$. This is a contradiction. The proof is complete.

The following identity plays a crucial part in the proof of Theorem 1.1.

**Lemma 2.3.** Let $u \equiv u(r, \alpha)$ and let $w$ be the solution of (2.3). Then
\begin{equation}
[r^{N-1}K^{-\frac{1}{2}}[w'u' - wu'] - r^{N-1}(K^{-\frac{1}{2}})'wu']' = -\frac{r^{N-2}}{4K^{\frac{5}{2}}}
\left[3r^2(K')^2 - 2r^2KK' + 2(N-1)rKK' + 4(N-1)K^2\right]w \frac{u'}{r},
\end{equation}
for $0 < r \leq 1$.

**Proof.** A direct calculation shows that (2.8) follows immediately.

**Remark 2.1.** We note that
\begin{equation}
 u''(0, \alpha) = \lim_{r \to 0^+} \frac{u'(r, \alpha)}{r} = -\frac{K(0)f(\alpha)}{N},
\end{equation}
and hence, the right side of (2.8) is continuous for $0 \leq r \leq 1$. In fact, by integrating (2.1) over $[0, r]$, we see that
\begin{equation*}
 u'(r, \alpha) = -r^{-(N-1)} \int_0^r t^{N-1}K(t)f(u(t, \alpha)) \, dt, \quad r \in [0, 1],
\end{equation*}
so that
\begin{align*}
 -\frac{r}{N} \max_{t \in [0, r]} K(t)f(u(t, \alpha)) &\leq u'(r, \alpha) \leq -\frac{r}{N} \min_{t \in [0, r]} K(t)f(u(t, \alpha)), \quad r \in [0, 1].
\end{align*}
Then we obtain (2.9).
Lemma 2.4. Assume that (1.4) holds. Then the solution $w$ of (2.3) has at least one zero in $(t_i, t_{i+1})$ for each $i \in \{1, 2, \ldots, k-1\}$.

Proof. Suppose that $w(r) \neq 0$ for $r \in (t_i, t_{i+1})$. We may assume that $w(r) > 0$ for $r \in (t_i, t_{i+1})$, since the case where $w(r) < 0$ for $r \in [t_i, t_{i+1}]$ can be treated similarly. Then we have $w(t_i) \geq 0$, $w(t_{i+1}) > 0$. In view of (1.1) we have

$$u''(t_j) = -K(t_j) f(u(t_j)), \quad j = 2, 3, \ldots, k.$$  

From (2.4) and (2.9) it follows that $(-1)^j u''(t_j) > 0$ for $j = 1, 2, \ldots, k$. Consequently we have

$$(-1)^j (-g(t_{i+1})w(t_{i+1})u''(t_{i+1}) + g(t_i)w(t_i)u''(t_i)) > 0,$$

where $g(r) = r^{N-1}|K(r)|^{-\frac{1}{2}}$. On the other hand, integrating (2.8) over $[t_i, t_{i+1}]$ and using (1.4) and (2.5), we find that

$$(-1)^j (-g(t_{i+1})w(t_{i+1})u''(t_{i+1}) + g(t_i)w(t_i)u''(t_i)) \leq 0.$$

This is a contradiction. The proof is complete.

Lemma 2.5. Let $w$ be the solution of (2.3). Assume that (1.3) and (1.4) hold. Then $(-1)^i w(z_i) < 0$ for $i = 1, 2, \ldots, k$.

Proof. Lemma 2.1 implies that $w(z_1) > 0$. By Lemmas 2.1 and 2.4, there exists a number $c_1 \in (z_1, t_2]$ such that $w(r) > 0$ for $r \in [0, c_1)$ and $w(c_1) = 0$. Then Lemma 2.2 implies that $w(r) < 0$ for $r \in (c_1, z_2]$. Hence we have $w(z_2) < 0$. From Lemma 2.4 it follows that there exists a number $c_2 \in (z_2, t_3]$ such that $w(r) < 0$ for $r \in (c_1, c_2)$ and $w(c_2) = 0$. By Lemma 2.2 we see that $w(r) > 0$ for $r \in (c_2, z_3]$, so that $w(z_3) > 0$. By continuing this process, we conclude that $(-1)^i w(z_i) < 0$ for $i = 1, 2, \ldots, k$. The proof is complete.
3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. To this end we employ the Prüfer transformation for the solution \( u(r, \alpha) \) of problem (1.1)-(2.2). For the solution \( u(r, \alpha) \) with \( \alpha > 0 \), we define the functions \( \rho(r, \alpha) \) and \( \theta(r, \alpha) \) by

\[
\begin{align*}
\rho(r, \alpha) & = r^{N-1}u'(r, \alpha), \\
\theta(r, \alpha) & = \frac{u(r, \alpha)}{r^{N-1}u'(r, \alpha)},
\end{align*}
\]

where \( \rho(r, \alpha) \) and \( \theta(r, \alpha) \) are written in the forms

\[
\begin{align*}
\rho(r, \alpha) & = \left( [u(r, \alpha)]^2 + r^{2(N-1)}[u'(r, \alpha)]^2 \right)^{1/2} > 0, \\
\theta(r, \alpha) & = \arctan \frac{u(r, \alpha)}{r^{N-1}u'(r, \alpha)}.
\end{align*}
\]

Therefore, since \( u(r, \alpha) \) and \( u'(r, \alpha) \) cannot vanish simultaneously, \( \rho(r, \alpha) \) and \( \theta(r, \alpha) \) are written in the forms

\[
\begin{align*}
\rho(r, \alpha) & = \left( [u(r, \alpha)]^2 + r^{2(N-1)}[u'(r, \alpha)]^2 \right)^{1/2} > 0, \\
\theta(r, \alpha) & = \arctan \frac{u(r, \alpha)}{r^{N-1}u'(r, \alpha)}.
\end{align*}
\]

For simplicity we take \( \theta(0, \alpha) = \pi/2 \). By a simple calculation we see that

\[
\theta'(r, \alpha) = \frac{1}{r^{N-1}} \cos^2 \theta(r, \alpha) + r^{N-1}K(r) \frac{\sin \theta(r, \alpha)f(\rho(r, \alpha)\sin \theta(r, \alpha))}{\rho(r, \alpha)} > 0
\]

for \( r \in (0, 1] \), which shows that \( \theta(r, \alpha) \) is strictly increasing in \( r \in (0, 1] \) for each fixed \( \alpha > 0 \). It is easy to see that \( u(r, \alpha) \) is a solution of \((P_k)\) if and only if

\[
\theta(1, \alpha) = k\pi,
\]

Hence the number of solutions of \((P_k)\) is equal to the number of roots \( \alpha > 0 \) of \((3.1)\).

Proposition 3.1. Let \( k \in \mathbb{N} \) and let \( u(r, \alpha_0) \) be a solution of \((P_k)\) for some \( \alpha_0 > 0 \). Suppose that \((1.3)\) and \((1.4)\) hold. Then \( \theta(1, \alpha_0) < 0 \).

Proof. Observe that

\[
\theta(1, \alpha_0) = \frac{u(1, \alpha_0)}{u'(1, \alpha_0)}
\]

Since \( u(1, \alpha_0) = 0 \) and \( z_k = 1 \), we obtain

\[
\theta(1, \alpha_0) = \frac{u(z_k, \alpha_0)}{u'(z_k, \alpha_0)}.
\]

Note that \((-1)^k u'(z_k, \alpha_0) > 0\), because of \((2.6)\). From Lemma 2.5, it follows that \((-1)^k u(\alpha_0) < 0\), which implies that \( \theta(1, \alpha_0) < 0 \). The proof is complete.

Proof of Theorem 1.1. Assume to the contrary that there exist numbers \( \alpha_1 > 0 \) and \( \alpha_2 > 0 \) such that \( u(r, \alpha_1) \) and \( u(r, \alpha_2) \) are solutions of \((P_k)\) and \( \alpha_1 \neq \alpha_2 \). Then \( \theta(1, \alpha_1) = \theta(1, \alpha_2) = k\pi \). We may assume without loss of generality that \( 0 < \alpha_1 < \alpha_2 \) and \( \theta(1, \alpha) \neq k\pi \) for \( \alpha \in (\alpha_1, \alpha_2) \). In view of Proposition 3.1, we
conclude that $\theta_{\alpha}(1, \alpha_{1}) < 0$ and $\theta_{\alpha}(1, \alpha_{2}) < 0$. The intermediate value theorem implies that there is a number $\alpha_{0} \in (\alpha_{1}, \alpha_{2})$ such that $\theta(1, \alpha_{0}) = k\pi$. This is a contradiction. Consequently, $(P_{k})$ has at most one solution. The proof of Theorem 1.1 is complete.

REFERENCES