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Kyoto University
Tridiagonal pairs and
the quantum affine
algebra $U_q(sl_2)$

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University of Wisconsin
Madison Wisconsin

Overview

In this talk, I will first recall the notion of a Leonard pair and discuss how these objects
are related to certain classical orthogonal polynomials.

I will then define a generalization of a Leonard pair called a Tridiagonal pair.

I will then show how certain tridiagonal pairs are related to finite dimensional modules for
the quantum affine algebra $U_q(sl_2)$.

Leonard pairs

We recall the notion of a Leonard pair. To do this, we first recall what it means for a matrix
to be tridiagonal.

The following matrices are tridiagonal.

\[
\begin{pmatrix}
2 & 3 & 0 & 0 \\
1 & 4 & 2 & 0 \\
0 & 5 & 3 & 3 \\
0 & 0 & 3 & 0
\end{pmatrix}, \quad \begin{pmatrix}
2 & 3 & 0 & 0 \\
0 & 4 & 2 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 1 & 5
\end{pmatrix}
\]

Tridiagonal means each nonzero entry lies on
either the diagonal, the subdiagonal, or the
superdiagonal.

The tridiagonal matrix on the left is irreducible.
This means each entry on the subdiagonal is
nonzero and each entry on the superdiagonal
is nonzero.

The Definition of a Leonard Pair

We now define a Leonard pair. From now on $\mathbb{K}$ will denote a field.

Definition Let $V$ denote a vector space over $\mathbb{K}$
with finite positive dimension. By a Leonard pair on $V$, we mean a pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ which satisfy both conditions below.

1. There exists a basis for $V$ with respect
to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

2. There exists a basis for $V$ with respect to
which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.
Example of a Leonard pair

For any nonnegative integer $d$ the pair

$$A = \begin{pmatrix} 0 & d & 0 & 0 \\ 1 & 0 & d-1 & 0 \\ 2 & & \ddots & \vdots \\ 0 & & & 1 \\ d & 0 & & \\ \end{pmatrix}$$

$A^* = \text{diag}(d, d-2, d-4, \ldots, -d)$

is a Leonard pair on the vector space $\mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is 0 or an odd prime greater than $d$.

Reason: There exists an invertible matrix $P$ such that $P^{-1}AP = A^*$ and $P^2 = 2^d I$.

Why Leonard pairs are of interest

There is a natural correspondence between Leonard pairs and a family of orthogonal polynomials consisting of the following types:

- $q$-Racah,
- $q$-Hahn,
- dual $q$-Hahn,
- $q$-Krawtchouk,
- dual $q$-Krawtchouk,
- quantum $q$-Krawtchouk,
- affine $q$-Krawtchouk,
- Racah,
- Hahn,
- dual-Hahn,
- Krawtchouk,
- Bannai/Ito,
- orphans (char($\mathbb{K}$) = 2 only).

This family coincides with the terminating branch of the Askey scheme of orthogonal polynomials.

Definition of a Tridiagonal pair

We say the pair $A, A^*$ is a Tridiagonal pair on $V$ whenever (1)–(4) hold below.

1. Each of $A, A^*$ is diagonalizable on $V$.

2. There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of $A$ such that

   $$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

   where $V_{-1} = 0$, $V_{d+1} = 0$.

3. There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of $A^*$ such that

   $$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

   where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

4. There is no subspace $W \subseteq V$ such that

   $$A W \subseteq W$$
   and $A^* W \subseteq W$ and $W \neq 0$ and $W \neq V$. 

Tridiagonal pairs

We now consider a generalization of a Leonard pair called a tridiagonal pair.

A tridiagonal pair is defined as follows.

As before, $V$ will denote a vector space over $\mathbb{K}$ with finite positive dimension.

As before, we consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$. 

5
A comment
Referring to our definition of a tridiagonal pair,
it turns out \( d = \delta \); we call this the **diameter** of the pair.

Leonard pairs and Tridiagonal pairs

We mentioned a tridiagonal pair is a generalization of a Leonard pair.

A Leonard pair is the same thing as a tridiagonal pair for which the eigenspaces \( V_i \) and \( V_i^* \) all have dimension 1.

An open problem

**Problem** Classify the tridiagonal pairs.

For the rest of this talk we focus on a special case of tridiagonal pair said to have **geometric type**.

We will show these tridiagonal pairs are related to \( U_q(\mathfrak{sl}_2) \).

We hope this will lead to a classification of the tridiagonal pairs of geometric type.

Tridiagonal pairs of geometric type

Let \( A, A^* \) denote a tridiagonal pair on \( V \) with diameter \( d \).

Let the eigenspaces \( V_i, V_i^* \) \( (0 \leq i \leq d) \) be as in the definition.

Let \( q \) denote a nonzero scalar in \( K \) which is not a root of unity.

We say \( A, A^* \) has **q-geometric type** whenever for \( 0 \leq i \leq d \), the eigenvalue of \( A \) for \( V_i \) is \( q^{2i-d} \) and the eigenvalue of \( A^* \) for \( V_i^* \) is \( q^{d-2i} \).
Tridiagonal pairs of geometric type and $U_q(sl_2)$

For the rest of this talk, $A, A^*$ denotes a tridiagonal pair on $V$ of diameter $d$ and $q$-geometric type.

Using $A, A^*$ we will construct two actions of $U_q(sl_2)$ on $V$.

We use the following notation.

Six Decompositions of $V$

We are about to define six decompositions of $V$.

In order to keep track of these decompositions we will give each of them a name.

Our naming scheme is as follows.

Let $\Omega$ denote the set consisting of the four symbols $0, D, 0^*, D^*$.

Each of the six decompositions will get a name $[u]$ where $u$ is a two-element subset of $\Omega$.

We now define the six decompositions.

Decompositions of $V$

By a decomposition of $V$, we mean a sequence $U_0, U_1, \ldots, U_d$ consisting of nonzero subspaces of $V$ such that

$$V = U_0 + U_1 + \cdots + U_d$$

(direct sum).

We do not assume each of $U_0, U_1, \ldots, U_d$ has dimension 1.

Six Decompositions of $V$, cont.

Lemma For each of the six rows in the table below, and for $0 \leq i \leq d$, let $V_i$ denote the $i$th subspace described in that row. Then the sequence $U_0, U_1, \ldots, U_d$ is a decomposition of $V$.

<table>
<thead>
<tr>
<th>name</th>
<th>$i$th subspace of the decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0D]$</td>
<td>$V_1$</td>
</tr>
<tr>
<td>$[0*D]$</td>
<td>$V_1^*$</td>
</tr>
<tr>
<td>$[0*]$</td>
<td>$(V_0 + \cdots + V_1^*) \cap (V_0 + \cdots + V_d)$</td>
</tr>
<tr>
<td>$[D*]$</td>
<td>$(V_0 + \cdots + V_1) \cap (V_0 + \cdots + V_d)$</td>
</tr>
<tr>
<td>$[D*D]$</td>
<td>$(V_0 + \cdots + V_1^*) \cap (V_0 + \cdots + V_d)$</td>
</tr>
<tr>
<td>$[D*]$</td>
<td>$(V_0 + \cdots + V_1) \cap (V_0 + \cdots + V_d)$</td>
</tr>
</tbody>
</table>
How the six decompositions are related

Let $U_0, U_1, \ldots, U_d$ denote any one of the six decompositions of $V$. Then for $0 \leq i \leq d$ the sums $U_0 + \cdots + U_i$ and $U_i + \cdots + U_d$ are given as follows.

<table>
<thead>
<tr>
<th>name</th>
<th>$U_0 + \cdots + U_i$</th>
<th>$U_i + \cdots + U_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0D]$</td>
<td>$V_0 + \cdots + V_i$</td>
<td>$V_i + \cdots + V_d$</td>
</tr>
<tr>
<td>$[O^<em>D^</em>]$</td>
<td>$V_0^* + \cdots + V_i^*$</td>
<td>$V_i^* + \cdots + V_d^*$</td>
</tr>
<tr>
<td>$[O^*O]$</td>
<td>$V_0^* + \cdots + V_i^*$</td>
<td>$V_i^* + \cdots + V_d^*$</td>
</tr>
<tr>
<td>$[D^*O]$</td>
<td>$V_{d-i}^* + \cdots + V_d^*$</td>
<td>$V_0 + \cdots + V_{d-i}$</td>
</tr>
<tr>
<td>$[D^*D]$</td>
<td>$V_{d-i} + \cdots + V_d$</td>
<td>$V_i + \cdots + V_d$</td>
</tr>
</tbody>
</table>

We now define four linear transformations from $V$ to $V$.

We call these $B, B^*, K, K^*$.

Each of these transformations is diagonalizable.

To define them, we list their eigenspaces and the corresponding eigenvalues as follows.

<table>
<thead>
<tr>
<th>name</th>
<th>$i$th subspace is eigenspace</th>
<th>Corresp. eigval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$[O^*O]$</td>
<td>$q^{a-d}$</td>
</tr>
<tr>
<td>$B^*$</td>
<td>$[D^*D]$</td>
<td>$q^{2a-2d}$</td>
</tr>
<tr>
<td>$K$</td>
<td>$[O^*D]$</td>
<td>$q^{2a-d}$</td>
</tr>
<tr>
<td>$K^*$</td>
<td>$[D^*O]$</td>
<td>$q^{2a-d}$</td>
</tr>
</tbody>
</table>

The $q$-Weyl Relations

We have

\[
\frac{qAB - q^{-1}BA}{q - q^{-1}} = I, \\
\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} = I, \\
\frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} = I, \\
\frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} = I.
\]

The $q$-Weyl Relations, cont.

We have

\[
\frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} = I, \\
\frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} = I, \\
\frac{qKA^* - q^{-1}A^*K}{q - q^{-1}} = I, \\
\frac{qB^*K - q^{-1}KB^*}{q - q^{-1}} = I.
\]
The $q$-Serre relations

We have

\[
A^3A - [3]_q A^2A^*A + [3]_q A A^*A^2 - A^*A^3 = 0,
\]
\[
A^*A^3 - [3]_q A^*A^2A^* + [3]_q A A^*A^2 - AA^*A^3 = 0,
\]
\[
B^3B^* - [3]_q B^2B^*B + [3]_q BB^*B^2 - B^*B^3 = 0,
\]
\[
B^*B^3 - [3]_q B^*B^2B^* + [3]_q B^*B^2B - BB^*B^3 = 0,
\]

where

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 0, 1, \ldots
\]

The algebra $U_q(\widehat{sl}_2)$

**Definition**

The quantum affine algebra $U_q(\widehat{sl}_2)$ is the unital associative $K$-algebra with generators $e_i^\pm, K_i^\pm, i \in \{0, 1\}$ and the following relations:

\[
K_i K_j^{-1} = K_j^{-1} K_i = 1,
\]
\[
K_0 K_1 = K_1 K_0,
\]
\[
K_i e_j^\pm K_i^{-1} = q^{\mp 2} e_j^\pm, \quad i \neq j,
\]
\[
K_i e_i^\pm K_i^{-1} = q^{\mp 2} e_i^\pm, \quad i \neq j,
\]
\[
[e_i^+, e_j^-] = \frac{K_i - K_j^{-1}}{q - q^{-1}},
\]
\[
[e_0^+, e_1^-] = 0.
\]

We call $e_i^\pm, K_i^\pm, i \in \{0, 1\}$ the Chevalley generators for $U_q(\widehat{sl}_2)$. 

Tridiagonal pairs of geometric type and $U_q(\widehat{sl}_2)$

We now use $A, A^*, B, B^*, K, K^*$ to get two actions of $U_q(\widehat{sl}_2)$ on $V$. Before proceeding we recall the definition of $U_q(\widehat{sl}_2)$. 

q-Weyl Relations cont.

We have

\[
\frac{qAK^*-q^{-1}K^*A}{q-q^{-1}} = I,
\]
\[
\frac{qK^*-1B-q^{-1}BK^*-1}{q-q^{-1}} = I,
\]
\[
\frac{qA^*K^{-1}-q^{-1}K^{-1}A^*}{q-q^{-1}} = I,
\]
\[
\frac{qK^*B^*-q^{-1}B^*K^*}{-1} = I,
\]

\[
\frac{qAK^* - q^{-1}K^*A}{q-q^{-1}} = I,
\]
\[
\frac{qK^*-1B-q^{-1}BK^*-1}{q-q^{-1}} = I,
\]
\[
\frac{qA^*K^{-1}-q^{-1}K^{-1}A^*}{q-q^{-1}} = I,
\]
\[
\frac{qK^*B^*-q^{-1}B^*K^*}{-1} = I.
\]
Another presentation of $U_q(\overline{sl_2})$

In order to state our main results we introduce a second presentation of $U_q(\overline{sl_2})$.

Two actions for $U_q(\overline{sl_2})$

**Theorem** Let $A, A^*$ denote a tridiagonal pair on $V$ of geometric type. Let the maps $B, B^*, K, K^*$ be as above.

Then $V$ is an Irreducible $U_q(\overline{sl_2})$-module on which the alternate generators act as follows.

<table>
<thead>
<tr>
<th>generator</th>
<th>$y_0^+$</th>
<th>$y_1^+$</th>
<th>$y_0^-$</th>
<th>$y_1^-$</th>
<th>$k_0$</th>
<th>$k_1$</th>
<th>$k_0^{-1}$</th>
<th>$k_1^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>action on $V$</td>
<td>$B^*$</td>
<td>$B$</td>
<td>$A^*$</td>
<td>$A$</td>
<td>$K$</td>
<td>$K^{-1}$</td>
<td>$K^{-1}$</td>
<td>$K$</td>
</tr>
</tbody>
</table>

Also, $V$ is an Irreducible $U_q(\overline{sl_2})$-module on which the alternate generators act as follows.

<table>
<thead>
<tr>
<th>generator</th>
<th>$y_0^+$</th>
<th>$y_1^+$</th>
<th>$y_0^-$</th>
<th>$y_1^-$</th>
<th>$k_0$</th>
<th>$k_1$</th>
<th>$k_0^{-1}$</th>
<th>$k_1^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>action on $V$</td>
<td>$A$</td>
<td>$A^*$</td>
<td>$B^*$</td>
<td>$B$</td>
<td>$K^*$</td>
<td>$K^{-2}$</td>
<td>$K^{-2}$</td>
<td>$K^{-1}$</td>
</tr>
</tbody>
</table>

In conclusion

From the previous theorem we readily obtain the following.

**Corollary** Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension.

Let $A, A^*$ denote a tridiagonal pair on $V$ of geometric type.

Then there exists a unique $U_q(\overline{sl_2})$-module structure on $V$ such that $y_1^-$ acts as $A$ and $y_0^+$ acts as $A^*$.

Moreover there exists a unique $U_q(\overline{sl_2})$-module structure on $V$ such that $y_0^+$ acts as $A$ and $y_1^+$ acts as $A^*$.

Both $U_q(\overline{sl_2})$-module structures are irreducible.

THE END