<table>
<thead>
<tr>
<th>Title</th>
<th>INTEGRABLE MODULES OVER $\widehat{\mathfrak{gl}}_m$ AND THE DOUBLE AFFINE HECKE ALGEBRA (Combinatorial Methods in Representation Theory and their Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Suzuki, Takeshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1438: 37-49</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47505">http://hdl.handle.net/2433/47505</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
INTEGRABLE MODULES OVER $\hat{\mathfrak{gl}}_m$ AND THE DOUBLE AFFINE HECKE ALGEBRA

京都大学数理解析研究所・鈴木 武史 (Takeshi Suzuki)
Research Institute for Mathematical Sciences,
Kyoto University

Introduction

Motivated by conformal field theory on the Riemann sphere, we introduce a certain space of coinvariants obtained from tensor product of representations of the affine Lie algebra $\hat{\mathfrak{gl}}_m$.

In [AST], an action of the degenerate affine Hecke algebra $H_\kappa$ is defined on this space through the Knizhnik-Zamolodchikov connection. This construction gives a functor from the category of highest (or lowest) weight modules over $\hat{\mathfrak{gl}}_m$ to the category of $H_\kappa$-modules.

We will see that the integrable $\hat{\mathfrak{gl}}_m$-modules correspond by this functor to irreducible $H_\kappa$-modules whose structure is described combinatorially. We also focus on the symmetric part of these irreducible $H_\kappa$-modules; i.e., the subspace consisting of those elements which are invariant with respect to the action of the Weyl group. We present a spectral decomposition of the symmetric part, and a character formula, which is described by level restricted analogue of the Kostka polynomial.

1. AFFINE LIE ALGEBRA

Throughout this note, we use the notation $[i, j] = \{i, i+1, \ldots, j\}$ for $i, j \in \mathbb{Z}$.

Let $m \in \mathbb{Z}_{\geq 2}$. Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{gl}_m$ consisting of all $n \times n$-matrices over $\mathbb{C}$. Let $\mathfrak{g}[t, t^{-1}]$ denote the Lie algebra consisting of all $n \times n$-matrices over $\mathbb{C}[t, t^{-1}]$. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c_\mathfrak{g}$ be the affine Lie algebra with the commutation relation

$$[a \otimes t^i, b \otimes t^j] = [a, b] \otimes t^{i+j} + \text{trace}(ab)i\delta_{i+j,0}c_\mathfrak{g}$$

for $a, b \in \mathfrak{g}$, $i, j \in \mathbb{Z}$.

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices, and let $\mathfrak{h}^*$ denote its dual space. A Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c_\mathfrak{g}$. Its dual space is denoted by $\hat{\mathfrak{h}}^*$. We regard $\hat{\mathfrak{h}}^*$ as a subspace of $\hat{\mathfrak{h}}^*$ through the identification $\hat{\mathfrak{h}}^* \cong \mathfrak{h}^* \oplus \mathbb{C}c_\mathfrak{g}^*$. 
Fix \( \ell \in \mathbb{C} \). For \( \lambda \in \mathfrak{h}^* \), \( \widehat{M}_{\ell}(\lambda) \) denote the highest weight Verma module of highest weight \( \lambda + \ell c_{\mathfrak{g}}^* \in \widehat{\mathfrak{h}}^* \), and let \( \widehat{M}_{\ell}^+(\lambda) \) denote the lowest weight Verma module of lowest weight \( -\lambda - \ell c_{\mathfrak{g}}^* \in \widehat{\mathfrak{h}}^* \). Their irreducible quotients are denoted by \( \widehat{L}_{\ell}(\lambda) \) and \( \widehat{L}_{\ell}^+(\lambda) \) respectively.

A \( \mathfrak{g} \)-module \( M \) is said to be of level \( \ell \) if \( c \) acts as a scalar \( \ell \). For example, \( \widehat{M}_{\ell}(\lambda) \) and \( \widehat{L}_{\ell}(\lambda) \) are of level \( \ell \), and \( \widehat{M}_{\ell}^+(\lambda) \) and \( \widehat{L}_{\ell}^+(\lambda) \) are of level \(-\ell \).

We identify \( \mathfrak{h} \) with \( \mathbb{C}^m \), and introduce its subspaces \( X_m = \mathbb{Z}^m \) and

\[
X_m^+ = \{ (\lambda_1, \ldots, \lambda_m) \in X_m \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \},
\]

\[
X_m^+(\ell) = \{ (\lambda_1, \ldots, \lambda_m) \in X_m^+ \mid \lambda_1 - \lambda_m \leq \ell \}.
\]

Note that \( \widehat{L}_{\ell}(\lambda) \) and \( \widehat{L}_{\ell}^+(\lambda) \) are integrable for \( \lambda \in X_m^+(\ell) \), and that \( X_m^+(\ell) \) is empty unless \( \ell \in \mathbb{Z}_{\geq 0} \).

Let \( E = \mathbb{C}^m \) denote the vector representation of \( \mathfrak{g} \). Put \( E[z, z^{-1}] = E \otimes \mathbb{C}[z, z^{-1}] \), which we regard as a \( \mathfrak{g}[t, t^{-1}] \)-module through the correspondence \( a \otimes t^k \mapsto a \otimes z^k \).

2. The degenerate double affine Hecke algebra

Let \( n \in \mathbb{Z}_{\geq 2} \). Let \( V \) denote the \( n \)-dimensional vector space over \( \mathbb{C} \) with the basis \( \{ y_i \}_{i \in [1,n]} \): \( V = \oplus_{i \in [1,n]} \mathbb{C}y_i \). Introduce the non-degenerate symmetric bilinear form \( \langle \mid \rangle \) on \( V \) by \( \langle y_i | y_j \rangle = \delta_{ij} \). Let \( V^* = \oplus_{i=1}^n \mathbb{C}x_i \) be the dual space of \( V \), where \( x_i \) is the dual vector of \( y_i \). The natural pairing is denoted by \( \langle \mid \rangle : V^* \times V \to \mathbb{C} \).

Put \( \alpha_{ij} = x_i - x_j \), \( \alpha_{ij}^\vee = y_i - y_j \) and \( \alpha_i = \alpha_{i_{i+1}} \). Then \( R = \{ \alpha_{ij} \mid i, j \in [1,n], i \neq j \} \) and \( R^+ = \{ \alpha_{ij} \in R \mid i < j \} \) give a set of roots and a set of positive roots of type \( A_{n-1} \) respectively.

Let \( W \) denote the Weyl group associated with the root system \( R \), which is isomorphic to the symmetric group \( \mathfrak{S}_n \) of degree \( n \). Denote by \( s_\alpha \) the reflection in \( W \) corresponding to \( \alpha \in R \). We write \( s_i = s_{\alpha_i} \) and \( s_{ij} = s_{\alpha_{ij}} \).

Put \( P = \oplus_{i \in [1,n]} \mathbb{Z}x_i \), which is preserved by \( W \). Define the extended affine Weyl group \( \overline{W} \) as the semidirect product \( P \rtimes W \) with the relation \( w\tau_{\eta}w^{-1} = \tau_{w(\eta)} \), where \( \tau_{\eta} \) denotes the element of \( \overline{W} \) corresponding to \( \eta \in P \).

Let \( S(V) \) denote the symmetric algebra of \( V \), which can be identified with the polynomial ring \( \mathbb{C}[y] = \mathbb{C}[y_1, \ldots, y_n] \).

Fix \( \kappa \in \mathbb{C} \). The degenerate double affine Hecke algebra (degenerate DAHA) \( H_\kappa \) of \( GL_n \) is an associative \( \mathbb{C} \)-algebra generated by the algebra \( \mathbb{C}P \), \( CW \) and \( S(V) \), and subjects to the following defining relations.
([C1]):

\[ s_i h = s_i(h)s_i - \langle \alpha_i | h \rangle \quad (i \in [1, n], \ h \in V), \]
\[ s_i e^\eta s_i = e^{s_i(\eta)} \quad (i \in [1, n], \ \eta \in P), \]
\[ [h, e^\eta] = \kappa \eta \langle \eta | h \rangle e^\eta + \sum_{\alpha \in R^+} \langle \alpha | h \rangle \frac{(e^\eta - e^{s_\alpha(\eta)})}{1 - e^{-\alpha}} s_\alpha \quad (h \in V, \ \eta \in P), \]

where \( e^\eta \) denote the element of \( \mathbb{C}P \) corresponding to \( \eta \in P \).

It is known that \( H_\kappa \cong \mathbb{C}P \otimes \mathbb{C}W \otimes S(V) \) as a vector space. The subalgebra \( H^{\text{aff}} = \mathbb{C}W \cdot S(V) \) is called the degenerate affine Hecke algebra. Note that the subalgebra \( \mathbb{C}P \cdot \mathbb{C}W \) is isomorphic to \( \mathbb{C}\overline{W} \).

3. INDUCED REPRESENTATIONS OF \( H_\kappa \)

For \( \lambda \in X_m = \mathbb{Z}^m \) we write \( \lambda \models n \) when \( \sum_{i \in [1,m]} \lambda_i = n \) and \( \lambda_i \in \mathbb{Z} \geq 0 \) for all \( i \in [1, m] \). Let \( \lambda, \mu \in X_m \) such that \( \lambda - \mu \models n \).

Introduce the subalgebra \( H_\lambda = \mathbb{C}W_{\lambda - \mu} \cdot S(V) \) of \( H_\kappa \), where \( W_{\lambda - \mu} \) denote the parabolic subgroup \( \mathfrak{S}_{\lambda_1 - \mu_1} \times \cdots \times \mathfrak{S}_{\lambda_m - \mu_m} \) of \( W \).

Let \( \mathbb{C}1_{\lambda, \mu} \) denote the one dimensional representation of \( H_{\lambda - \mu} \) such that

\[ w1_{\lambda, \mu} = 1_{\lambda, \mu} \quad (w \in W_{\lambda - \mu}), \]
\[ y_i 1_{\lambda, \mu} = \langle \zeta_{\lambda, \mu} | y_i \rangle 1_{\lambda, \mu} \quad (i \in [1, n]), \]

where \( \zeta_{\lambda, \mu} \) denote the element of \( V^* \) given by

\[ \langle \zeta_{\lambda, \mu} | y_i \rangle = \mu_j + i - m_j - j - 1 \quad \text{for} \quad i \in [m_j + 1, m_{j+1}], \]

with \( m_0 = 0 \) and \( m_j = \sum_{k \in [1,j]} (\lambda_k - \mu_k) \) \( (j \in [1, m]) \). Define an \( H_\kappa \)-module by \( \mathcal{M}_\kappa(\lambda, \mu) = H_\kappa \otimes_{H_{\lambda - \mu}} \mathbb{C}1_{\lambda, \mu} \). Obviously we have

\[ \mathcal{M}_\kappa(\lambda, \mu) \cong \mathbb{C}\overline{W}/W_{\lambda - \mu} \cong \mathbb{C}P \otimes \mathbb{C}W/W_{\lambda - \mu} \]

as an \( \overline{W} \)-module.

In the rest, we often identify the group ring \( \mathbb{C}P \) with the Laurent polynomial ring \( \mathbb{C}[z^\pm] = \mathbb{C}[z_1^\pm, \ldots, z_n^\pm] \) via the correspondence \( e^{x_i} \mapsto z_i \).

**Example 3.1.** Let \( m = 1 \) and let \( \lambda = (n) \) and \( \mu = (0) \). Then \( \mathcal{M}_\kappa(\lambda, \mu) \cong \mathbb{C}P = \mathbb{C}[z^\pm] \), which is called the (Laurent) polynomial representation. On the representation \( \mathbb{C}P \), the element \( y_i \) \( (i \in [1, n]) \) acts as the Cherednik-Dunkl operator

\[ T_i = \kappa z_i \frac{\partial}{\partial z_i} + \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - s_\alpha) + \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - s_\alpha) + i - 1. \]
The simultaneous eigenvectors of $T_1, \ldots, T_n$ are called the nonsymmetric Jack polynomials.

4. THE SPACE OF COINVARIANTS AND THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

Let $\ell \in \mathbb{C}$. Let $M$ be a highest weight module of level $\ell$ and let $N$ be a lowest weight module of level $-\ell$. We set

$$\tilde{C}(M, N) = M \otimes E[z_1, z_1^{-1}] \otimes \cdots \otimes E[z_n, z_n^{-1}] \otimes N,$$

$$C(M, N) = \tilde{C}(M, N)/\mathfrak{g}[t, t^{-1}]\tilde{C}(M, N).$$

Let $\sigma_{ij} \in \text{End}_\mathbb{C}[z^{\pm 1}]$ denote the permutation of $z_i$ and $z_j$. Let $\Omega_{ij} \in \text{End}_\mathbb{C}(E^\otimes n)$ denote the permutation of $i$-th and $j$-th component of the tensor product. Note that $\tilde{C}(M, N) \cong M \otimes E^\otimes n \otimes \mathbb{C}[z^{\pm 1}] \otimes N$, through which we regard $\sigma_{ij}$ and $\Omega_{ij}$ as elements in $\text{End}_\mathbb{C}(\tilde{C}(M, N))$.

For $i \in [0, n+1]$, define $\theta_i : \mathfrak{h} \to U(\mathfrak{g})^\otimes n+2$ by $\theta_i(u) = 1^\otimes i \otimes u \otimes 1^\otimes n-i+1$. For $i, j \in [0, n+1]$ with $i < j$, define $\theta_{ij} : \mathfrak{h}^\otimes 2 \to U(\mathfrak{g})^\otimes n+2$ by $\theta_{ij}(u \otimes v) = 1^\otimes i \otimes u \otimes 1^\otimes j-i-1 \otimes v \otimes 1^\otimes n-j+1$.

Define the linear operators on $\tilde{C}(M, N)$ by

$$D_i = \kappa z_i \frac{\partial}{\partial z_i} + \tilde{r}_{0i} - \tilde{r}_{in+1} + \sum_{1 \leq j<i} r_{ij} - \sum_{i<j \leq n} r_{ji} + \theta_i(\rho^\vee) + \sum_{1 \leq j<i} \frac{z_j}{z_i - z_j} (1 - \sigma_{ij}) \Omega_{ij} + \sum_{i<j \leq n} \frac{z_i}{z_i - z_j} (1 - \sigma_{ij}) \Omega_{ij} + i - 1,$$

where $\rho^\vee = \sum_{k \in [1, m]} \frac{1}{2}(n-2k+1)e_{aa} \in \mathfrak{h}$.

**Theorem 4.1** (Theorem 4.2.2 in [AST]). Let $M$ be a highest weight module of $\mathfrak{h}$ of level $\kappa - m$ and let $N$ be a lowest weight module of level $-\kappa + m$. 
The following statement has been shown in [AST].

**Proposition 5.1** (Proposition 5.3.1 in [AST]). Let $\kappa \in \mathbb{C}$ and put $\ell = \kappa - m$.

(i) Let $\lambda, \mu \in X_m^+$. Then

$$C(\mathcal{M}_{\ell}(\mu), \mathcal{M}_{\ell}^+(\lambda)) \cong \begin{cases} \mathcal{M}_{\kappa}(\lambda, \mu) & \text{if } \lambda - \mu \models n, \\ 0 & \text{otherwise}. \end{cases}$$

(ii) Let $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then

$$C(\mathcal{L}_{\ell}(\mu), \mathcal{L}_{\ell}^+(\lambda)) \cong C(\mathcal{M}_{\ell}(\mu), \mathcal{L}_{\ell}^+(\lambda)) \cong C(\hat{L}_{\ell}(\mu), \hat{L}_{\ell}^+(\lambda)).$$

For each $\lambda \in X_m$, we have the additive functor $F_{\lambda}(-) = C(-, \mathcal{M}_{\ell}(\lambda))$ from the category of highest weight modules over $\hat{\mathfrak{g}}$ to the category of $H_{\kappa}$-modules. It is right exact and sends the Verma module $\mathcal{M}_{\ell}(\mu)$ to the induced module $\mathcal{M}_{\kappa}(\lambda, \mu)$ by Proposition 5.1. In the sequel, we will determine the image $F_{\lambda}(\hat{L}_{\ell}(\mu))$ of the irreducible module $\hat{L}_{\ell}(\mu)$ in the case where $\lambda, \mu \in X_m^+(\ell)$. Note that $F_{\lambda}(\hat{L}_{\ell}(\mu)) \cong C(\hat{L}_{\ell}(\mu), \hat{L}_{\ell}^+(\lambda))$, and note also that it is a quotient of $F_{\lambda}(\mathcal{M}_{\kappa}(\lambda, \mu))$.

Let $\ell \in \mathbb{Z}_{\geq 0}$ and $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then it is known that the $H_{\kappa}$-module $\mathcal{M}_{\kappa}(\lambda, \mu)$ has a unique simple quotient ([AST, S1]), which we will denote by $\mathcal{L}_{\kappa}(\lambda, \mu)$.

The irreducible modules $\mathcal{L}_{\kappa}(\lambda, \mu)$ for $\lambda, \mu \in X_m^+(\ell)$ are investigated in [SV], and in particular their structure is described combinatorially using tableaux on periodic skew diagrams. We give a short review of the theory of periodic tableaux and the tableaux representations of $H_{\kappa}$ in Appendix. By means of this combinatorial description, we can estimate the kernel of the projection $\mathcal{M}_{\kappa}(\lambda, \mu) \rightarrow \mathcal{L}_{\kappa}(\lambda, \mu)$. By comparing it with the kernel of $\mathcal{M}_{\kappa}(\lambda, \mu) \rightarrow F_{\lambda}(\hat{L}_{\ell}(\mu))$, we have...
Theorem 5.2. Let $\kappa \in \mathbb{Z}_{\geq 1}$ and put $\ell = \kappa - m$. Let $\lambda, \mu \in X_{m}^{+}(\ell)$ such that $\lambda - \mu \vdash n$. Then the $H_{\kappa}$-module $C(\widehat{L}_{\ell}(\mu), \widehat{L}_{\ell}^{\dagger}(\lambda))$ is irreducible:

$$C(\widehat{L}_{\ell}(\mu), \widehat{L}_{\ell}^{\dagger}(\lambda)) \cong \mathcal{L}_{\kappa}(\lambda, \mu),$$

and moreover it is semisimple over $S(V)$. (See Theorem A.3 for the combinatorial description of the weight decomposition).

The classification of the irreducible $H_{\kappa}$-modules which are semisimple over $S(V)$ is given in [C2, SV], from which (or from Theorem A.4) we have

Corollary 5.3. Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $L$ be an irreducible $H_{\kappa}$-module which is finitely generated and admits a weight decomposition of the form $L = \oplus_{\zeta \in P} L_{\zeta}$, where $L_{\zeta} = \{v \in L \mid yv = \langle \zeta \mid y \rangle \forall y \in V\}$. Then there exists $m \in [1, n]$ and $\lambda, \mu \in X_{m}^{+}(\kappa - m)$ such that $L \cong C(\widehat{L}_{\kappa-m}(\mu), \widehat{L}_{\kappa-m}^{\dagger}(\lambda))$.

6. Localization and Conformal Coinvariants

We will see the relation between our space $C(M, N)$ of coinvariants and the space of conformal coinvariants in Wess-Zumino-Witten model [TK, TUY].

Observe that the group ring $\mathbb{C}P$ can be seen as the coordinate ring $A = \mathbb{C}[T]$ of the affine variety $T = (\mathbb{C} \setminus \{0\})^{n}$. Put $T_{o} = T \setminus \Delta$, where $\Delta = \cup_{i<j}\{\xi_{1}, \ldots, \xi_{n} \in T \mid \xi_{i}/\xi_{j} = 1\}$, and put $A_{o} = \mathbb{C}[T_{o}]$. Namely, $A_{o}$ is the localization of $A$ at $\Delta$; $A_{o} = \mathbb{C}\left[z_{i}^{\pm 1}, \ldots, z_{n}^{\pm 1}, \frac{1}{1-z_{i}/z_{j}} \left(i \neq j\right)\right]$. Let $D(T_{o})$ denote the ring of algebraic differential operators on $T_{o}$. Then the Cherednik-Dunkl operators in Example 3.1 $T_{1}, \ldots, T_{n}$ can be seen as elements of the ring $D(T_{o}) \times \mathbb{C}W$. Put $H_{\kappa, o} = A_{o} \otimes_{A} H_{\kappa}$. There exists a unique algebra structure on $H_{\kappa, o}$ extending $H_{\kappa}$.

Proposition 6.1. Let $\kappa \in \mathbb{C}^{\times}$. There exists a unique algebra isomorphism $H_{\kappa, o} \rightarrow D(T_{o}) \times \mathbb{C}W$ such that $y_{i} \mapsto T_{i}$, $w \mapsto w$, $f \mapsto f$ for all $i \in [1, n]$, $w \in W$ and $f \in A_{o}$.

For an $H_{\kappa}$-module $M$, set $M_{o} = A_{o} \otimes_{A} M$. Then via Proposition 6.1, we have a structure of $D(T_{o}) \times \mathbb{C}W$-module on $M_{o}$; namely, $M_{o}$ admits a $W$-equivariant integrable (algebraic) connection

$$\nabla_{i} = \kappa^{-1}\left\{y_{i} - \sum_{1 \leq j < i} \frac{z_{j}}{z_{i} - z_{j}}(1 - s_{\alpha}) - \sum_{i < j \leq n} \frac{z_{i}}{z_{i} - z_{j}}(1 - s_{\alpha}) - (i - 1)\right\}.$$ 

Now consider the case where $M = C(\widehat{L}_{\ell}(\mu), \widehat{L}_{\ell}^{\dagger}(\lambda)) = \mathcal{L}_{\kappa}(\lambda, \mu)$ with $\lambda, \mu \in X_{m}^{+}(\ell)$. Then it follows that the connection given above has regular singularities along $\Delta$, and hence $\mathcal{L}_{\kappa}(\lambda, \mu)_{o}$ is a projective $A_{o}$-module, or geometrically, a vector bundle over $T_{o}$ of finite rank ([GGOR, VV]).
For $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{T}_\circ$, let $\mathbb{C}_\xi$ denote the one-dimensional right module of $A_\circ$ given by the evaluation at $\xi$. It follows that the space $\mathbb{C}_\xi \otimes_{A_\circ} (L_\kappa(\lambda, \mu)_0)$ is isomorphic to with "the space of conformal coinvariants"

$$\left( \hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1)^{\otimes n} \otimes \hat{L}_\ell(\lambda^\dagger) \right) / \mathfrak{g}(0, \xi, \infty) \left( \hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1)^{\otimes n} \otimes \hat{L}_\ell(\lambda^\dagger) \right),$$

where $\nu_1 = (1, 0, \ldots, 0) \in X_m^+(\ell)$ (the highest weight of the vector representation $E$), $\lambda^\dagger = -w_0(\lambda)$ with $w_0$ being the longest element of $W$, and $\mathfrak{g}(0, \xi, \infty)$ denotes the Lie algebra of $\mathfrak{g}$-valued algebraic functions on $\mathbb{P}^1 \setminus \{0, \xi_1, \ldots, \xi_n, \infty\}$, which acts on $\hat{L}_\ell(\mu) \otimes \hat{L}_\ell(\nu_1)^{\otimes n} \otimes \hat{L}_\ell(\lambda^\dagger)$ through the Laurent expansion at each points. (See e.g. [BK] for a precise definition.)

Therefore it follows that the vector bundle $L_\kappa(\lambda, \mu)_0$ is equivalent to the vector bundle of conformal coinvariants (the dual of the vector bundle of conformal blocks in the sense of [TUY, BK]). Moreover, the connection $\{\nabla_i\}$ on $L_\kappa(\lambda, \mu)_0$ given via Proposition 6.1 coincides with the Knizhnik-Zamolodchikov connection on the vector bundle of conformal coinvariants.

7. WEIGHT DECOMPOSITION OF SYMMETRIC PART

For an $H_\kappa$-module $M$, put

$$M^W = \{v \in M \mid wv = v \forall w \in W\},$$

(7.1)

on which the algebra $H_\kappa^W = \{u \in H_\kappa \mid wuw^{-1} = u\}$ acts. The algebra $H_\kappa^W$ is called the zonal spherical algebra and it contains a subalgebra $S(V)^W$, which coincides with the center of the degenerate affine Hecke algebra $H^{\text{aff}}$.

For $\zeta \in V^*$, let $\chi_\zeta$ denote the image of the projection to the quotient space $W \setminus V^*$. Identify $W \setminus V^*$ with the set $\text{Hom}_{\text{algebra}}(S(V)^W, \mathbb{C})$ of characters, and set

$$M^W_{[\zeta]} = \{v \in M^W \mid \xi v = \chi_\zeta(\xi) v \forall \xi \in S(V)^W\}.$$

In the sequel, we will give a decomposition of $L_\kappa(\lambda, \mu)^W$ into weight spaces with respect to $S(V)^W$.

Let $\lambda, \mu \in X_m^+$ such that $\lambda - \mu \vdash n$. Let $\lambda/\mu$ denote the skew Young diagram associated with $(\lambda, \mu)$:

$$\lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1, m], b \in [\mu_a + 1, \lambda_a]\}.$$

(7.2)

Let $T$ be a tableau on the diagram $\lambda/\mu$; namely $T$ is a bijection from $\lambda/\mu$ to $[1, n]$. Then it determines the sequence $\{\lambda_T^{(i)}\}_{i \in [0, n]}$ in $X_m$ by the condition $\lambda_T^{(0)} = \mu$ and $\lambda_T^{(i)}/\lambda_T^{(i-1)} = T^{-1}(i)$ ($i \in [1, n]$).
Let $\ell \in \mathbb{Z}_{\geq 0}$. A tableau $T$ is called an $\ell$-restricted standard tableau if $\lambda_T^{(i)} \in X_m^+(\ell)$ for all $i \in [1, n]$. Let $\text{St}(\ell)(\lambda, \mu)$ denote the set of $\ell$-restricted tableaux on $\lambda$.

Let $T \in \text{St}(\ell)(\lambda, \mu)$. For $i \in [1, n]$, define

$$(7.3) \quad h_i(T) = \begin{cases} 1 & \text{if } a < a', \\ 0 & \text{if } a \geq a', \end{cases}$$

where $T(a,b) = i$ and $T(a',b') = i + 1$. Define

$$(7.4) \quad \eta_T = \sum_{i \in [1,n]} \left( \sum_{j < i} h_j(T) \right) x_i \in P.$$ 

Define $\zeta_T \in V^*$ by $\zeta_T(y_i) = b - a$ when $T(a,b) = i$.

From the weight decomposition of $\mathcal{L}_\kappa(\lambda, \mu)$ (Theorem A.3) with respect to $S(V)$, we have

**Theorem 7.1. (Conjecture 6.1.1 in [AST])** Let $\mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then

$$\mathcal{L}_\kappa(\lambda, \mu)^W = \bigoplus_{\nu \in P^-} \bigoplus_{T \in \text{St}(\ell)(\lambda/\mu)} \mathcal{L}_\kappa(\lambda, \mu)_{[\zeta_T + \kappa(\nu + \eta_T)]}^W,$$

where $P^- = \{ \zeta \in P | \langle \zeta | \alpha_i^\vee \rangle \leq 0 \ \forall i \in [1, n-1] \}$, and

$$\dim \mathcal{L}_\kappa(\lambda, \mu)_{[\zeta_T + \kappa(\nu + \eta_T)]}^W = 1$$

for all $\nu \in P^-$ and $T \in \text{St}(\ell)(\lambda/\mu)$.

**8. q-DIMENSION FORMULA**

Put $\partial = \kappa^{-1} \sum_{i \in [1,n]} y_i \in S(V)^W$. Then $\partial$ satisfies the relation

$$[\partial, z_i] = \kappa z_i, \quad [\partial, w] = 0$$

for all $i \in [1, n]$ and $w \in W$.

Our next purpose is to give a $q$-dimension formula for $\mathcal{L}_\kappa(\lambda, \mu)^W$ with respect to the grading operator $\partial$. To this end, we need to introduce the “polynomial part” of $\mathcal{L}_\kappa(\lambda, \mu)$ following [AST].

Define a subalgebra $H^0_\kappa$ of $H_\kappa$ by

$$H^0_\kappa = \mathbb{C}P^0 \cdot \mathbb{C}W \cdot S(V),$$

where $P^0 = \bigoplus_{i \in [1,n]} \mathbb{Z}_{\geq 0} x_i$.

Let $\kappa \in \mathbb{Z}_{\geq 1}$ and let $\mu \in X_m^+(\kappa - m)$ such that $\lambda - \mu \models n$. Recall that the induced module $\mathcal{M}_\kappa(\lambda, \mu)$ is generated by the cyclic vector $1_{\lambda, \mu}$. We denote by $1_{\lambda, \mu}^{-}$ its image under the projection $\mathcal{M}_\kappa(\lambda, \mu) \rightarrow \mathcal{L}_\kappa(\lambda, \mu)$. Note that $1_{\lambda, \mu} \neq 0$. Define the polynomial part of $\mathcal{L}_\kappa(\lambda, \mu)$ by $\mathcal{L}^0_\kappa(\lambda, \mu) = H^0_\kappa 1_{\lambda, \mu}$, which is an $H^0_\kappa$-submodule of $\mathcal{L}_\kappa(\lambda, \mu)$. 
Put $\mathcal{L}_{\kappa}^{\geq 0}(\lambda, \mu)^{W}_{(k)} = \{v \in \mathcal{L}_{\kappa}^{\geq 0}(\lambda, \mu)^{W} \mid \partial v = kv\}$. Then we have \(\dim \mathcal{L}_{\kappa}^{\geq 0}(\lambda, \mu)^{W}_{(k)} < \infty\) and $\mathcal{L}_{\kappa}^{\geq 0}(\lambda, \mu)^{W} = \oplus_{k \in \mathbb{Z}} \mathcal{L}_{\kappa}^{\geq 0}(\lambda, \mu)^{W}_{(k)}$. Define

\[
\dim_q \mathcal{L}_{\kappa}^{\geq 0}(\lambda, \mu)^{W} = \sum_{d \in \mathbb{Z}} q^k \dim \mathcal{L}_{\kappa}^{\geq 0}(\lambda, \mu)^{W}_{(k)}.
\]

Set

\[
(8.1) \quad h(T) = \kappa \langle \eta_T \mid \partial \rangle = \sum_{i \in [1,n]} (n-i)h_i(T).
\]

From Theorem 7.1, we have

**Theorem 8.1.** Let $\kappa \in \mathbb{Z}_{\geq 0}$ and let $\lambda, \mu \in X_+^+(\kappa-m)$ such that $\lambda - \mu \models n$. Then

\[
(8.2) \quad \dim_q \mathcal{L}_{\kappa}^{\geq 0}(\lambda, \mu)^{W} = \frac{q^{\Delta_{\lambda}-\Delta_{\mu}}}{(q)_n} F_{\lambda/\mu}^{(\ell)}(q).
\]

Here $\Delta_{\lambda} = \frac{1}{2\kappa}((\lambda, \lambda) + 2(\rho, \lambda))$, $(q)_n = (1-q)(1-q^2) \ldots (1-q^n)$ and $F_{\lambda/\mu}^{(\ell)}(q)$ is a polynomial of $q$ given by

\[
(8.3) \quad F_{\lambda/\mu}^{(\ell)}(q) = \sum_{T \in \text{St}_{(\ell)}(\lambda/\mu)} q^{h(T)}.
\]

**Remark 8.2.** If $\ell$ is large enough then $F_{\lambda/\mu}^{(\ell)}(q)$ coincides with the Kostka polynomial $K_{\lambda/\mu}'(1^n)(q)$ associated to the conjugate $(\lambda/\mu)'$ of $\lambda/\mu$. Hence our polynomial $F_{\lambda/\mu}^{(\ell)}(q)$ is an $\ell$-restricted version of the Kostka polynomial (cf. [FJKLM]).

**Remark 8.3.** A bosonic formula for $F_{\lambda/\mu}^{(\ell)}(q)$ is known (Theorem 6.2.4 in [AST]), and Theorem 8.1 is equivalent to the formula in Conjecture 6.1.1 in [AST]. Note also that the bosonic formula suggests the existence of the BGG type resolution of $\mathcal{L}_{\kappa}(\lambda, \mu)$.

9. **RATIONAL ANALOGUE**

For a $\mathfrak{g}[t]$-module $N$, set

\[
\tilde{C}(M) = E[z_1] \otimes \cdots \otimes E[z_n] \otimes N
\]

\[
C(N) = \tilde{C}(N)/\mathfrak{g}[t]\tilde{C}(N),
\]

where $E[z] = E \otimes \mathbb{C}[z]$. The analogous construction gives on $C(N)$ an action of the rational Cherednik algebra $H_{\kappa}^{\text{rat}}([EG])$, which can be
defined as the subalgebra of $H_{\kappa}$ generated by the subalgebra $\mathbb{C}[z] \cdot \mathbb{C}W$ and the following (pairwise commutative) elements

$$u_{i} = z_{i}^{-1} \left( y_{i} - \sum_{j<i} s_{ij} \right) \quad (i \in [1,n])$$

as pointed out in [S2].

It follows for $\lambda \in X_{m}^{+}(\ell)$ that $C(\overline{M}^{\dagger}_{\ell}(\lambda))$ is isomorphic to some induced module, and $C(\overline{L}^{\dagger}_{\ell}(\lambda))$ is isomorphic to the unique simple quotient of $C(\overline{M}^{\dagger}_{\ell}(\lambda))$, which we denote by $L_{\kappa}(\lambda)$.

Let $0 = (0, \ldots, 0) \in X_{m}^{+}(\ell)$. Then it follows that the polynomial part $L_{\kappa}^{\geq 0}(\lambda, 0)$ of the $H_{\kappa}$-module $L_{\kappa}(\lambda, 0)$ is an $H_{\kappa}^{\text{rat}}$-submodule and it is isomorphic to $L_{\kappa}(\lambda)$. This leads the $q$-dimension formula

$$\dim_{q} L_{\kappa}(\lambda)^{W} = \frac{q^{\Delta_{\lambda}}}{(q)_{n}} F_{\lambda}^{(\ell)}(q).$$

**Remark 9.1.** It can be seen that the Knizhnik-Zamolodchikov functor investigated in [GGOR] transforms the irreducible representations $L_{\kappa}(\lambda)$ for $A \in X_{m}^{+}(\ell)$ to Wenzl’s representations [W] of the affine Hecke algebra (cf. [TK]).

**Appendix A. Tableaux on periodic diagrams and representations of the degenerate DAHA**

We will review the theory of tableaux representations for $H_{\kappa}$, which is investigated in [SV] for the double affine Hecke algebra.

Fix $\kappa \in \mathbb{Z}_{\geq 1}$. Let $m \in \mathbb{Z}_{\geq 1}$.

For $\lambda, \mu \in X_{m}^{+}(\kappa - m)$ such that $\lambda - \mu \models n$, we introduce the following subsets of $\mathbb{Z} \times \mathbb{Z}$:

\begin{itemize}
    \item[(A.1)] $\lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1,m], \ b \in [\mu_{a} + 1, \lambda_{a}]\}$,
    \item[(A.2)] $\overline{\lambda/\mu} = \{(a, b) + k(m, -\kappa + m) \in \mathbb{Z} \times \mathbb{Z} \mid (a, b) \in \lambda/\mu, \ k \in \mathbb{Z}\}$.
\end{itemize}

The set $\overline{\lambda/\mu}$ is called the periodic skew diagram of period $(m, -\kappa + m)$ associated with $(\lambda, \mu)$. The following is called the skew property:

**Lemma A.1.** Let $(a, b), (a', b') \in \overline{\lambda/\mu}$. If $a' - a \in \mathbb{Z}_{\geq 0}$ and $b' - b \in \mathbb{Z}_{\geq 0}$ then $(a, b'), (a', b) \in \overline{\lambda/\mu}$.

A tableau $T$ on $\overline{\lambda/\mu}$ is by definition a bijection $\overline{\lambda/\mu} \to \mathbb{Z}$ satisfying $T(a + m, b - \kappa + m) = T(a, b) + n$ for all $(a, b) \in \overline{\lambda/\mu}$.

A tableau $T$ is called a standard tableau if

$$T(a, b + 1) > T(a, b)$$
for any \((a, b), (a, b + 1) \in \lambda/\mu\), and if
\[ T(a + 1, b) > T(a, b) \]
for any \((a, b), (a + 1, b) \in \lambda/\mu\). Let \(\text{Tab}(\lambda/\mu)\) and \(\text{St}(\lambda/\mu)\) denote the set of tableaux and the set of standard tableaux on \(\lambda/\mu\) respectively.

Define the elements \(\pi = \tau_{x_{1}}s_{1}s_{2}\cdots s_{n-1}\) and \(s_{0} = \tau_{\alpha_{1n}}s_{1n}\) of the group \(\overline{W} = P \rtimes W\). Then \(\{s_{0}, s_{1}, \ldots, s_{n-1}, \pi\}\) is a generator of the group \(\overline{W}\).

Define the action of \(\overline{W}\) on the set \(\mathbb{Z}\) of integers by
\[
\begin{align*}
s_{i}(j) &= \begin{cases} j + 1 & \text{for } j \equiv i \text{ mod } n, \\ j - 1 & \text{for } j \equiv i + 1 \text{ mod } n, \\ j & \text{for } j \not\equiv i, i + 1 \text{ mod } n, \end{cases} \\
t_{x_{i}}(j) &= \begin{cases} j + n & \text{for } j \equiv i \text{ mod } n, \\ j & \text{for } j \not\equiv i \text{ mod } n. \end{cases}
\end{align*}
\]
Observe that \(\pi(j) = j + 1\) for all \(j\).

For \(T \in \text{Tab}(\lambda/\mu)\) and \(w \in \overline{W}\), the map \(wT : \lambda/\mu \rightarrow \mathbb{Z}\) given by
\[
(wT)(u) = w(T(u)) \quad (u \in \lambda/\mu)
\]
is also a tableau on \(\lambda/\mu\), and the assignment \(T \mapsto wT\) gives an action of \(\overline{W}\) on \(\text{Tab}(\lambda/\mu)\), which preserves \(\text{St}(\lambda/\mu)\). It is easy to see that the assignment \(w \mapsto wT\) gives a one-to-one correspondence \(\overline{W} \cong \text{Tab}(\lambda/\mu)\).

Define the map \(C : \lambda/\mu \rightarrow \mathbb{Z}\) by \(C(a, b) = b - a\), and define \(C_T : \mathbb{Z} \rightarrow \mathbb{Z}\) by \(C_T(i) = C(T^{-1}(i))\) for \(T \in \text{St}(\lambda/\mu)\). Define \(\zeta_T \in V^*\) by \(\langle \zeta_T | y_i \rangle = C_T(i) (i \in [1, n])\).

The following lemma follows from the skew property and the definition of the standard tableaux:

**Lemma A.2.** Let \(T \in \text{St}(\lambda/\mu)\) and \(i \in [0, n - 1]\).

(i) \(C_T(i) - C_T(i + 1) \neq 0\).

(ii) \(s_i T \in \text{St}(\lambda/\mu)\) if and only if \(C_T(i) - C_T(i + 1) \notin \{-1, 1\}\).

Now, we introduce the tableaux representation associated with \(\lambda/\mu\).

Let \(V_\kappa(\lambda/\mu)\) be the vector space with the basis \(\{v_T\}_{T \in \text{St}(\lambda/\mu)}\): \[
V_\kappa(\lambda/\mu) = \bigoplus_{T \in \text{St}(\lambda/\mu)} \mathbb{C}v_T.
\]

By Lemma A.2 and induction argument, we have
Theorem A.3. (Theorem 3.16, Theorem 3.17 in [SV]) Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $\lambda, \mu \in X_{m}^{+}(\kappa - m)$ such that $\lambda - \mu \models n$.

(i) There exists a unique $H_{\kappa}$-module structure on $\mathcal{V}_{\kappa}(\overline{\lambda}/\mu)$ such that

$$y_{i}v_{T} = C_{T}(i)v_{T} \quad (i \in [1, n]),$$
$$\pi v_{T} = v_{\pi T},$$
$$s_{i}v_{T} = \begin{cases} 
\frac{1+a_{i}}{a_{i}}v_{s_{i}T} - \frac{1}{a_{i}}v_{T} & \text{if } s_{i}T \in \text{St}(\overline{\lambda}/\mu) \\
-\frac{1}{a_{i}}v_{T} & \text{if } s_{i}T \notin \text{St}(\overline{\lambda}/\mu) 
\end{cases} \quad (i \in [0, n - 1]),$$

where $a_{i} = C_{T}(i) - C_{T}(i + 1) \neq 0$ (by Lemma A.2).

(ii) $\mathcal{V}_{\kappa}(\lambda, \mu) = \bigoplus_{T \in \text{St}(\overline{\lambda}/\mu)} \mathcal{V}_{\kappa}(\lambda, \mu)_{\xi_{T}}$, and $\mathcal{V}_{\kappa}(\lambda, \mu)_{\xi_{T}} \cong \mathbb{C}v_{T}$ for all $T \in \text{St}(\overline{\lambda}/\mu)$.

(iii) The $H_{\kappa}$-module $\mathcal{V}_{\kappa}(\overline{\lambda}/\mu)$ is irreducible.

(iv) $\mathcal{V}_{\kappa}(\overline{\lambda}/\mu) \cong L_{\kappa}(\lambda, \mu)$.

The following result is also announced in [C2]:

Theorem A.4. (Theorem 3.19 in [SV]) Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $L$ be an irreducible $H_{\kappa}$-module such that $L = \bigoplus_{\xi \in \mathcal{P}} L_{\xi}$. Then there exist $m \in [1, n]$ and $\lambda, \mu \in X_{m}^{+}(\kappa - m)$ with $\lambda - \mu \models n$ such that $L \cong \mathcal{V}_{\kappa}(\overline{\lambda}/\mu)$.

REFERENCES


*E-mail address: takeshi@kurims.kyoto-u.ac.jp*