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Path Model for a Level-Zero Extremal Weight Module over a Quantum Affine Algebra

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0 Introduction.

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra over the field $\mathbb{Q}$ of rational numbers, and let $P$ be an integral weight lattice of $\mathfrak{g}$. In [L1] and [L2], Littelmann introduced the path model consisting of Lakshmibai-Seshadri paths (LS paths for short) for a representation of the symmetrizable Kac-Moody algebra $\mathfrak{g}$; for an integral weight $\lambda \in P$, an LS path of shape $A$ is, by definition, a path $\pi : [0, 1] \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} P$ (i.e., piecewise linear, continuous maps such that $\pi(0) = 0$ and $\pi(1) \in P$) determined by a pair of a sequence of elements in $\mathfrak{w} \lambda$, where $\mathfrak{w}$ is the Weyl group of $\mathfrak{g}$, and a sequence of rational numbers satisfying a certain combinatorial condition (see §1.2 below). We denote by $\mathcal{B}(\lambda)$ the set of all LS paths of shape $\lambda$. Littelmann showed that the set $\mathcal{B}(\lambda)$ together with root operators (see §1.3 below) and the weight map $\text{wt}(\pi) := \pi(1)$, $\pi \in \mathcal{B}(\lambda)$, is a crystal with weight lattice $P$. Then he proved that if $\lambda \in P$ is a dominant integral weight, then the crystal graph of the crystal $\mathcal{B}(\lambda)$ is connected, and the formal sum $\sum_{\pi \in \mathcal{B}(\lambda)} e(\pi(1))$ is equal to the character $\text{ch} L(\lambda)$ of the integrable highest weight $\mathfrak{g}$-module $L(\lambda)$ of highest weight $\lambda$. Moreover, it was proved independently by Kashiwara [Kas3] and Joseph [J] that the $\mathcal{B}(\lambda)$ for dominant $\lambda$ is, as a crystal, isomorphic to the crystal base of the highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of highest weight $\lambda$, where $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in $q$. Now, quite a natural question arises: Is there any $U_q(\mathfrak{g})$-module whose crystal base is isomorphic to the crystal $\mathcal{B}(\lambda)$ for general $\lambda \in P$? In a series of papers [NS1] ~ [NS3], we gave a kind of answer to this question in the case where $\mathfrak{g}$ is an affine Lie algebra.
For a more precise description, we need some notation. Let $\mathfrak{g}$ be an affine Lie algebra over $\mathbb{Q}$ with Cartan subalgebra $\mathfrak{h}$, simple roots $\{\alpha_j\}_{j \in I} \subset \mathfrak{h}^*$, simple coroots $\{h_j\}_{j \in I} \subset \mathfrak{h}$, and Weyl group $W = \langle r_j \mid j \in I \rangle \subset \text{GL}(\mathfrak{h}^*)$, where $r_j$, $j \in I$, are the simple reflections. We denote by $\delta = \sum_{j \in I} a_j \alpha_j \in \mathfrak{h}^*$ the null root, and by $c = \sum_{j \in I} a_j^\vee h_j \in \mathfrak{h}$ the canonical central element. An integral weight $\lambda \in P$ is said to be of positive (resp., negative) level if $\lambda(c) > 0$ (resp., $\lambda(c) < 0$), and to be of level zero if $\lambda(c) = 0$:

$$P = \left\{ \lambda \in P \mid \lambda(c) > 0 \right\} \cup \left\{ \lambda \in P \mid \lambda(c) = 0 \right\} \cup \left\{ \lambda \in P \mid \lambda(c) < 0 \right\}.$$ 

If $\lambda \in P$ is of positive (resp., negative) level, then there exists a unique dominant (resp., anti-dominant) integral weight in $W\lambda$. Denote it by $\mu$. Because $B(\lambda) = B(w\lambda)$ for all $w \in W$, we have that the set $B(\lambda)$ is the same as the set $B(\mu)$ of all LS paths of shape $\mu$; accordingly, it follows from the result due to Kashiwara [Kas3] and Joseph [J] that $B(\lambda)$ is, as a crystal, isomorphic to the crystal base of the highest (resp., lowest) weight module $V(\mu)$ of highest (resp., lowest) weight $\mu$ over the quantum affine algebra $U_q(\mathfrak{g})$.

Now we are left with the case where $\lambda \in P$ is of level zero. We take (and fix) a special vertex $0 \in I$ such that $a_0^\vee = 1$, and set $I_0 := I \setminus \{0\}$. Let $\Lambda_i$, $i \in I$, be the fundamental weights for $\mathfrak{g}$, and set $\varpi_i := \Lambda_i - a_i^\vee \Lambda_0$ for $i \in I_0$ (note that $\varpi_i$, $i \in I$, is a level-zero integral weight). In the case where $\lambda = m\varpi_i$ for some $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, we proved in [NS1] and [NS2] that the LS path crystal is isomorphic to the crystal base of the extremal weight module over $U_q(\mathfrak{g})$ (Theorem 1). Here the extremal weight module $V(\lambda)$ over $U_q(\mathfrak{g})$ with $\lambda$ as an extremal weight is an integrable module over $U_q(\mathfrak{g})$ generated by a single element $v_\lambda$ with the defining relations that the $v_\lambda$ is an extremal weight vector of weight $\lambda$ (see §1.4 below); we know from [Kas1, Proposition 8.2.2] that the extremal weight module $V(\lambda)$ admits a crystal base, denoted by $B(\lambda)$.

**Theorem 1.** For $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, the crystal $B(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal, isomorphic to the crystal base $B(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ with $m\varpi_i$ as an extremal weight.

We know from [NS1, Remark 5.2] and [NS3, §3.1] that for a general integral weight $\lambda \in P$ of level zero, there is no isomorphism of crystals between the set...
$\mathcal{B}(\lambda)$ of all LS paths of shape $\lambda$ and the crystal base $\mathcal{B}(\lambda)$ of the extremal weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of extremal weight $\lambda$. We do not know whether or not there exists a $U_q(\mathfrak{g})$-module having a crystal base isomorphic to $\mathcal{B}(\lambda)$, except for the case mentioned in Theorem 1.

Now we turn to a fundamental module of level zero (see §1.5 below). Let $\text{cl} : \mathfrak{h}^* \to \mathfrak{h}^*/\mathbb{Q}\delta$ be the canonical projection. Denote by $U'_q(\mathfrak{g})$ the quantized universal enveloping algebra with $P_{cl} := \text{cl}(P)$ the integral weight lattice. In [Kas4, §5.2], Kashiwara introduced a finite-dimensional irreducible $U'_q(\mathfrak{g})$-module $W(\varpi_i)$, called a fundamental module of level zero, and proved that it has a global basis with a simple crystal (see [Kas4, Theorem 5.17]). The fundamental module $W(\varpi_i)$ of level zero seems to be isomorphic to the Kirillov-Reshetikhin module $W^{(i)}_1$ in the notation of [HKOTT, §2.3] for $i \in I_0$ (see [HKOTT, Remark 2.3]). In [NS1] and [NS2], we gave a path model for $W(\varpi_i) \cong W^{(i)}_1$ as follows. Let $\lambda \in P$ be a level-zero integral weight. For an LS path $\pi \in \mathcal{B}(\lambda)$ of shape $\lambda$, we define a path $\text{cl}(\pi) : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P_{cl}$ by: $(\text{cl}(\pi))(t) = \text{cl}(\pi(t))$ for $t \in [0,1]$, and set $\mathcal{B}(\lambda)_{cl} := \text{cl}(\mathcal{B}(\lambda))$. Then the set $\mathcal{B}(\lambda)_{cl}$ has a crystal structure with weight lattice $P_{cl}$, which is naturally induced from that of $\mathcal{B}(\lambda)$.

**Theorem 2.** The crystal $\mathcal{B}(\varpi_i)_{cl}$ is isomorphic to the crystal base of the fundamental module $W(\varpi_i)$ of level zero.

In [NS3], we studied the crystal structure of $\mathcal{B}(\lambda)_{cl} = \text{cl}(\mathcal{B}(\lambda))$ for a general integral weight $\lambda \in P$ of level zero. Before stating our main result of [NS3], we make some comments. If $\lambda' = \lambda + R\delta$ for some $R \in \mathbb{Q}$, then it follows from the definition of LS paths that $\mathcal{B}(\lambda') = \{ \pi + R\delta : \pi \in \mathcal{B}(\lambda) \}$, where $(\pi + R\delta)(t) := \pi(t) + tR\delta$, $t \in [0,1]$, and from the definition of the root operators that the crystal graph of $\mathcal{B}(\lambda + R\delta)$ is the same shape as that of $\mathcal{B}(\lambda)$, up to $R\delta$-shift of weight. In addition, we have that $\mathcal{B}(\lambda) = \mathcal{B}(w\lambda)$ for all $w \in W$. Therefore we may assume that the $\lambda \in P$ is of the form $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$ from the beginning. Now we are ready to state our main result in [NS3].

**Theorem 3.** Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. Then, there exists a unique isomorphism $\mathcal{B}(\lambda)_{cl} \simeq \bigotimes_{i \in I_0} (\mathcal{B}(\varpi_i)_{cl})^{\otimes m_i}$ of crystals (with weight lattice $P_{cl}$) between the crystal $\mathcal{B}(\lambda)_{cl}$ and the tensor product $\bigotimes_{i \in I_0} (\mathcal{B}(\varpi_i)_{cl})^{\otimes m_i}$. 
By combining Theorems 2 and 3, we can get the following corollary.

**Corollary.** Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. Then, the crystal $B(\lambda)_{\text{cl}}$ is, as a crystal with weight lattice $P_{\text{cl}}$, isomorphic to the crystal base of the tensor product $U'_q(\mathfrak{g})$-module $\bigotimes_{i \in c_0} W(\varpi_i)^{\otimes m_i}$.

1 Preliminaries.

1.1 Affine Lie algebras and quantum affine algebras. Let $\mathfrak{g}$ be an affine Lie algebra over the field $\mathbb{Q}$ of rational numbers with Cartan subalgebra $\mathfrak{h}$. Denote by $\Pi := \{\alpha_j\}_{j \in I} \subset \mathfrak{h}^*$ the set of simple roots, and by $\Pi' := \{h_j\}_{j \in I} \subset \mathfrak{h}$ the set of simple coroots, where $I = \{0, 1, 2, \ldots, \ell\}$ is an index set for the simple roots $\Pi$. Throughout this article, we use the numbering of the simple roots as in [Kac, §4.8 and §6]. Let $\delta \in \mathfrak{h}^*$ and

$$c = \sum_{j \in I} a_j^\vee h_j \in \mathfrak{h}$$

be the null root and the canonical central element of $\mathfrak{g}$, respectively. Denote by $W = \langle r_j \mid j \in I \rangle \subset \text{GL}(\mathfrak{h}^*)$ the Weyl group of the affine Lie algebra $\mathfrak{g}$, where $r_j \in \text{GL}(\mathfrak{h}^*)$ is the simple reflection in $\alpha_j$ for $j \in I$. We call an element of the set $\Delta^\text{re} := W\Pi$ a real root, and denote by $\Delta^\text{re}_+$ the set of positive real roots. Let $\Lambda_j$, $j \in I$, be the fundamental weights for the affine Lie algebra $\mathfrak{g}$. We take (and fix) an integral weight lattice $P \subset \mathfrak{h}^*$ that contains all the simple roots $\alpha_j$, $j \in I$, and fundamental weights $\Lambda_j$, $j \in I$. For each $i \in I_0 := I \setminus \{0\}$, we define a level-zero fundamental weight $\varpi_i \in P$ by

$$\varpi_i := \Lambda_i - a_i^\vee \Lambda_0.$$  

Note that $\varpi_i(c) = 0$; an integral weight $\lambda \in P$ is said to be level-zero if $\lambda(c) = 0$. An integral weight $\lambda \in P$ of level zero is said to be dominant if $\lambda(h_i) \geq 0$ for all $i \in I_0$. Let

$$\text{cl} : \mathfrak{h}^* \rightarrow \mathfrak{h}^*/\mathbb{Q}\delta$$

be the canonical projection, and set $P_{\text{cl}} := \text{cl}(P)$.

Let $U'_q(\mathfrak{g})$ be the quantized universal enveloping algebra (with weight lattice $P$) of the affine Lie algebra $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in $q$. We
denote by $E_j$, $F_j$, $j \in I$, and $q^h$, $h \in P^\vee := \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$ the Chevalley generators of $U_q(\mathfrak{g})$, where $E_j$ (resp., $F_j$) corresponds to the simple root $\alpha_j$ (resp., $-\alpha_j$). Denote by $U'_q(\mathfrak{g})$ the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $E_j$, $F_j$, $j \in I$, and $q^h$, $h \in (P_0)^\vee := \text{Hom}_\mathbb{Z}(P_{\text{c}1}, \mathbb{Z})$, which is the quantized universal enveloping algebra of $\mathfrak{g}$ with weight lattice $P_{\text{c}1}$.  

1.2 Lakshmibai–Seshadri paths. A path (with weight in $P$) is, by definition, a piecewise linear, continuous map $\pi : [0, 1] \to \mathbb{Q} \otimes \mathbb{Z} P$ from $[0, 1] := \{ t \in \mathbb{Q} \mid 0 \leq t \leq 1 \}$ to $\mathbb{Q} \otimes \mathbb{Z} P$ such that $\pi(0) = 0$ and $\pi(1) \in P$. In this subsection, we recall the definition of a Lakshmibai–Seshadri path (an LS path for short) from [L2, §4] (see also [NS2, §1.4] and [NS3, §2.1]). 

We first recall some auxiliary notations. Let $\lambda \in P$ be an integral weight. For $\mu, \nu \in W\lambda$, we write $\mu \geq \nu$ if there exist a sequence $\mu = \xi_0, \xi_1, \ldots, \xi_n = \nu$ of elements in $W\lambda$ and a sequence $\beta_1, \ldots, \beta_n \in \Delta^+_{\mathbb{R}}$ of positive real roots such that $\xi_k = r_{\beta_k}(\xi_{k-1})$ and $\xi_{k-1}(\beta^\vee_k) < 0$ for $k = 1, 2, \ldots, n$, where for a positive real root $\beta \in \Delta^+_{\mathbb{R}}$, $r_\beta$ denotes the reflection with respect to $\beta$, and $\beta^\vee$ denotes the dual real root of $\beta$. If $\mu \geq \nu$, then we define $\text{dist}(\mu, \nu)$ to be the maximal length $n$ of all possible such sequences $\xi_0, \xi_1, \ldots, \xi_n$ for the pair $(\mu, \nu)$. Then, for $\mu, \nu \in W\lambda$ with $\mu > \nu$ and a rational number $0 < a < 1$, an $a$-chain for $(\mu, \nu)$ is, by definition, a sequence $\mu = \xi_0 > \xi_1 > \cdots > \xi_n = \nu$ of elements in $W\lambda$ such that $\text{dist}(\xi_{k-1}, \xi_k) = 1$ and $a\xi_{k-1}(\beta^\vee_k) \in \mathbb{Z}_{<0}$ for all $k = 1, 2, \ldots, n$, where $\beta_k$ is the positive real root corresponding to $(\xi_{k-1}, \xi_k)$ with $\xi_{k-1} > \xi_k$.

Now we are ready for the definition of an LS path. Let $\lambda \in P$ be an integral weight. An LS path of shape $\lambda$ is a path $\pi : [0, 1] \to \mathbb{Q} \otimes \mathbb{Z} P$ associated to a pair $(\nu; g)$ of a sequence $\nu : \nu_1, \nu_2, \ldots, \nu_s$ of elements in $W\lambda$ and a sequence $g : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers satisfying the condition that there exists an $a_k$-chain for $(\nu_k, \nu_{k+1})$ for all $k = 1, 2, \ldots, s - 1$; to such a pair $(\nu; g) = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s)$, we associate the following path $\pi : [0, 1] \to \mathbb{Q} \otimes \mathbb{Z} P$:

$$\pi(t) = \sum_{l=1}^{k-1} (a_l - a_{l-1}) \nu_l \, + \, (t-a_{k-1}) \nu_k \quad \text{for } a_{k-1} \leq t \leq a_k, \, 1 \leq k \leq s.$$ 

Note that $\pi(0)$ is obviously equal to $0 \in P$, and it follows from [L2, Lemma 4.5 a)]
that $\pi(1) \in P$; namely, the $\pi$ above is, in fact, a path for all such pairs $(\nu; a) = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s)$. Denote by $\mathcal{B}(\lambda)$ the set of LS paths of shape $\lambda$.

**Remark 1.2.1.** (1) The straight line $\pi_\nu(t) := t\nu$, $t \in [0, 1]$, is contained in $\mathcal{B}(\lambda)$ for all $\nu \in W\lambda$ (put $s = 1$ and $\nu_1 = \nu$).

(2) It follows from the definition that $\mathcal{B}(w\lambda) = \mathcal{B}(\lambda)$ for all $w \in W$.

**1.3 Root operators.** In this subsection, we give a description of root operators $e_j$ and $f_j$, $j \in I$, which was introduced in [L2, §1], on the set $\mathcal{B}(\lambda)$ of all LS paths of shape $\lambda \in P$ (see also [NS2, §1.2] and [NS4, §2.1]).

Let $\lambda \in P$ be an integral weight. For an LS path $\pi \in \mathcal{B}(\lambda)$ and $j \in I$, we define $e_j\pi$ as follows: First, we set

\[ H_j^\pi(t) := (\pi(t))(h_j) \quad \text{for} \quad t \in [0, 1], \]
\[ m_j^\pi := \min \{H_j^\pi(t) | t \in [0, 1]\}. \tag{1.3.1} \]

If $m_j^\pi > -1$, then we define $e_j\pi := \theta$. Here, $\theta$ is an extra element, which corresponds to the 0 in the theory of crystals (by convention, we put $e_j\theta = f_j\theta := \theta$).

If $m_j^\pi \leq -1$, then

\[
(e_j\pi)(t) := \begin{cases} 
\pi(t) & \text{if } 0 \leq t \leq t_0, \\
\pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \leq t \leq t_1, \\
\pi(t) + \alpha_j & \text{if } t_1 \leq t \leq 1,
\end{cases}
\tag{1.3.2}
\]

where we set

\[
t_1 := \min \{t \in [0, 1] \mid H_j^\pi(t) = m_j^\pi \}, \\
t_0 := \max \{t' \in [0, t] \mid H_j^\pi(t) \geq m_j^\pi + 1 \text{ for all } t \in [0, t'] \}.
\]

Similarly, $f_j\pi$ is given as follows: If $H_j^\pi(1) - m_j^\pi < 1$, then we set $f_j\pi := \theta$. If $H_j^\pi(1) - m_j^\pi \geq 1$, then

\[
(f_j\pi)(t) := \begin{cases} 
\pi(t) & \text{if } 0 \leq t \leq t_0, \\
\pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \leq t \leq t_1, \\
\pi(t) - \alpha_j & \text{if } t_1 \leq t \leq 1,
\end{cases}
\tag{1.3.3}
\]

where we set

\[
t_0 := \max \{t \in [0, 1] \mid H_j^\pi(t) = m_j^\pi \}, \\
t_1 := \min \{t' \in [t_0, 1] \mid H_j^\pi(t) \geq m_j^\pi + 1 \text{ for all } t \in [t', 1] \}.
\]
Theorem 1.3.1 ([L2]). For every integral weight $\lambda \in P$, the set $\mathcal{B}(\lambda) \cup \{\theta\}$ is stable under the action of the root operators $e_j$ and $f_j$ for $j \in I$. We define

$$
\begin{align*}
\text{wt}(\pi) & := \pi(1) \quad \text{for } \pi \in \mathcal{B}(\lambda), \\
e_j(\pi) & := \max\{n \geq 0 \mid e_j^n \pi \neq \theta\} \quad \text{for } \pi \in \mathcal{B}(\lambda) \text{ and } j \in I, \\
f_j(\pi) & := \max\{n \geq 0 \mid f_j^n \pi \neq \theta\} \quad \text{for } \pi \in \mathcal{B}(\lambda) \text{ and } j \in I.
\end{align*}
$$

Then, the set $\mathcal{B}(\lambda)$ together with the root operators and the maps above is a crystal with weight lattice $P$.

1.4 Extremal weight modules.

Definition 1.4.1 (cf. [Kas1, §8] and [Kas4, §3.1]). Let $M$ be an integrable $U_q(\mathfrak{g})$-module. A vector $v \in M$ of weight $\lambda \in P$ is said to be extremal, if there exists a family $\{v_w\}_{w \in W}$ of weight vectors of $M$ satisfying the following conditions: for $w \in W$ and $j \in I$,

a) $v_w = v$ if $w = 1$;

b) if $n := (\lambda)(h_j) \geq 0$, then $E_j v_w = 0$ and $E_j^{(n)} v_w = v_{r_j w}$;

c) if $n := (\lambda)(h_j) \leq 0$, then $F_j v_w = 0$ and $E_j^{(-n)} v_w = v_{r_j w}$.

Here, $E_j^{(n)}$ and $F_j^{(n)}$ are the $n$-th $q$-divided powers of the Chevalley generators $E_j$ and $F_j$ of $U_q(\mathfrak{g})$, respectively.

Definition 1.4.2 (cf. [Kas1, §8] and [Kas4, §3.1]). Let $\lambda \in P$ be an integral weight. The extremal weight module $V(\lambda)$ over $U_q(\mathfrak{g})$ with $\lambda$ as an extremal weight is, by definition, the integrable $U_q(\mathfrak{g})$-module generated by a single element $v_\lambda$ with the defining relations that $v_\lambda$ is an extremal vector of weight $\lambda$.

We know the following theorem from [Kas1, Proposition 8.2.2].

Theorem 1.4.3. For every $\lambda \in P$, the extremal weight module $V(\lambda)$ has a crystal base, which we denote by $\mathcal{B}(\lambda)$.

Remark 1.4.4. The extremal weight module is a natural generalization of an integrable highest and lowest weight module; in fact, we know from [Kas1, §8] that if $\lambda \in P$ is dominant (resp. anti-dominant), then the extremal weight module $V(\lambda)$ is isomorphic to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight $\lambda$, and the crystal base $\mathcal{B}(\lambda)$ of $V(\lambda)$ is isomorphic to the crystal base of the integrable highest (resp., lowest) weight module as a crystal.
1.5 Fundamental module of level zero. We define a positive integer $d_i \in \mathbb{Z}_{\geq 1}$ by

$$\{n \in \mathbb{Z} \mid \varpi_i + n\delta \in W\varpi_i\} = \mathbb{Z}d_i.$$  

(1.5.1)

Because $V(\varpi_i) \cong V(w\varpi_i)$ as $U_q(\mathfrak{g})$-modules for all $w \in W$ (see [Kas1, Proposition 8.2.2 iv]), we see that there exists a $U_q'(\mathfrak{g})$-module isomorphism $V(\varpi_i) \cong V(\varpi_i + d_i\delta)$, which maps the $\varpi_i$-weight space $V(\varpi_i)_{\varpi_i}$ of $V(\varpi_i)$ to the $(\varpi_i + d_i\delta)$-weight space $V(\varpi_i + d_i\delta)_{\varpi_i + d_i\delta}$ of $V(\varpi_i + d_i\delta)$ (by [Kas4, Proposition 5.16], these weight spaces are 1-dimensional). Thus we get a $U_q'(\mathfrak{g})$-module automorphism $z_i : V(\varpi_i) \cong V(\varpi_i)$ of weight $d_i\delta$ (see [Kas4, §5.2]) as the composition of these maps. We now define a $U_q'(\mathfrak{g})$-module $W(\varpi_i)$ by

$$W(\varpi_i) := V(\varpi_i)/(z_i - 1)V(\varpi_i),$$  

(1.5.2)

which is called a fundamental module of level zero. We know from [Kas4, Theorem 5.17] that $W(\varpi_i)$ is a finite-dimensional irreducible $U_q'(\mathfrak{g})$-module, and has a simple crystal base, which is denoted by $B(\varpi_i)_{\mathfrak{c}1}$.

2 Our results.

2.1 Isomorphism theorems. Our main result in [NS1] and [NS2] is the following theorem (see [NS1, Theorem 5.1] and [NS2, Corollaries 2.2.1 and 3.3.8]).

Theorem 2.1.1. For $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, the crystal $B(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal with weight lattice $P$, isomorphic to the crystal base $B(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ with $m\varpi_i$ as an extremal weight.

Here, let us give a sketch of our proof of Theorem 2.1.1. First we show the theorem for the case where $m = 1$. In [NS2, Theorem 2.1.1], we proved the following.

Theorem 2.1.2. For every $i \in I_0$, the crystal graph of the crystal $B(\varpi_i)$ is connected.
We know from [Kas4, Proposition 5.4(ii)] that the crystal graph of the crystal base $\mathcal{B}(\varpi_i)$ is also connected, and from [Kas4, Proposition 5.16(ii)] that the cardinality of the subset $\mathcal{B}(\varpi_{i})_{\varpi_{i}}$ is equal to 1 for all $w \in W$, where $\mathcal{B}(\varpi_{i})_\mu$ is the subset of $\mathcal{B}(\varpi_i)$ consisting of all elements of weight $\mu \in P$. In addition, we see from [BN, Theorem 4.16(i)] that there exists a canonical embedding $\mathcal{B}_0(N\varpi_i) \hookrightarrow \mathcal{B}(\varpi_i)^{\otimes N}$ of crystals that sends $u_{N\varpi_i}$ to $u_{\varpi_i}^{\otimes N}$, where for each $A \in P$, $u_{\lambda}$ denotes the element of the crystal base $\mathcal{B}(\lambda)$ corresponding to the generator $v_{\lambda}$ of the extremal weight module $V(\lambda)$, and $\mathcal{B}_0(\lambda)$ denotes the connected component of $\mathcal{B}(\lambda)$ containing the element $u_{\lambda}$. Further we showed the following proposition.

**Proposition 2.1.3 ([NS1, Theorem 3.1]).** For every $N \in \mathbb{Z}_{>0}$ and $i \in I_0$, there exists an injective map $S_N : \mathcal{B}(\varpi_i) \hookrightarrow \mathcal{B}_0(N\varpi_i)$, which we call an $N$-multiple map, satisfying the following condition:

1. $S_N(u_{\varpi_i}) = u_{N\varpi_i}$,
2. $\text{wt}(S_N(b)) = N \text{wt}(b)$ for each $b \in \mathcal{B}(\varpi_i)$,
3. $S_N(e_j b) = e_j^N S_N(b)$, $S_N(f_j b) = f_j^N S_N(b)$ for $b \in \mathcal{B}(\varpi_i)$ and $i \in I$.

By using these facts, we can show that $\mathcal{B}(\varpi_i) \cong \mathcal{B}(\varpi_i)$ as crystals in exactly the same way as [Kas2, Theorem 4.1] (see [NS1, Theorem 5.1]).

As a consequence of Theorem 2.1.1 for the case where $m = 1$, we obtained the following corollary (cf. [NS1, Corollary 5.3]).

**Corollary 2.1.4.** For every $m \geq 1$ and $i \in I_0$, we have

$$\mathcal{B}_0(m\varpi_i) \cong \mathcal{B}_0(m\varpi_i) \quad \text{as crystals},$$

where $\mathcal{B}_0(m\varpi_i)$ is the connected component of the crystal $\mathcal{B}(m\varpi_i)$ containing the straight line $\pi_{m\varpi_i}(t) = t(m\varpi_i)$, $t \in [0,1]$.

Next we prove Theorem 2.1.1 for the case where $m \geq 2$ (as seen below, the crystal graph of $\mathcal{B}(m\varpi_i)$ is not connected when $m \geq 2$). Let $\text{Par}_{<m}$ be the set of partitions of length (i.e., the number of parts) strictly less than $m$. For each $\sigma = (k_1 \geq k_2 \geq \cdots \geq k_{m-1}) \in \text{Par}_{<m}$, we denote by $|\sigma|$ the weight of $\sigma$, i.e.,
We can define a crystal structure on $\text{Par}_{<m}$ as follows:

\[
\begin{align*}
|\sigma| := k_1 + k_2 + \cdots + k_{m-1} \\
e_j \sigma = f_j \sigma = 0 & \quad \text{for all } \sigma \in \text{Par}_{<m} \text{ and } j \in I, \\
e_j(\sigma) = \varphi_j(\sigma) = 0 & \quad \text{for all } \sigma \in \text{Par}_{<m} \text{ and } j \in I, \\
\text{wt}(\sigma) := -|\sigma|d_0 & \quad \text{for } \sigma \in \text{Par}_{<m}.
\end{align*}
\]

In [NS2, §§3.2 ~ 3.6], we showed the following.

**Lemma 2.1.5.** (1) For every $\sigma = (k_1 \geq k_2 \geq \cdots \geq k_{m-1}) \in \text{Par}_{<m}$,

\[
\pi_{\sigma} := \left(m(\varpi_i - k_1 d_0), \ldots, m(\varpi_i - k_{m-1} d_0), m\varpi_i; 0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\right).
\]

is contained in $\mathbb{B}(m\varpi_i)$.

(2) For each $\pi \in \mathbb{B}(m\varpi_i)$, there exists a unique $\sigma \in \text{Par}_{<m}$ such that the $\pi$ is connected to $\pi_{\sigma}$ in the crystal graph of $\mathbb{B}(m\varpi_i)$.

For $\sigma \in \text{Par}_{<m}$, we denote by $\mathbb{B}_\sigma(m\varpi_i)$ the connected component of $\mathbb{B}(m\varpi_i)$ containing the path $\pi_{\sigma}$. Then it follows from the lemma above that

\[
\mathbb{B}(m\varpi_i) = \bigsqcup_{\sigma \in \text{Par}_{<m}} \mathbb{B}_\sigma(m\varpi_i).
\]

Here recall from §1.3 that the root operators $e_j, f_j$ are defined in terms of the function given by the pairing of a path and the simple coroot $h_j$. Because the path $\pi_{\sigma}(t)$ is the same as the straight line $\pi_{m\varpi_i}(t) = t(m\varpi_i)$, up to some $\delta$-shift, and because $\delta(h_j) = 0$ for all $j \in I$, we deduce that the crystal graph of $\mathbb{B}_0(m\varpi_i)$ is isomorphic to the crystal graph of $\mathbb{B}_\sigma(m\varpi_i)$, up to some $\delta$-shift of weight. More precisely, we have

\[
\mathbb{B}_\sigma(m\varpi_i) \cong \{\sigma\} \otimes \mathbb{B}_0(m\varpi_i) \hookrightarrow \text{Par}_{<m} \otimes \mathbb{B}_0(\varpi_i) \text{ as crystals,}
\]

which sends $\pi_{\sigma}$ to $\sigma \otimes \pi_{m\varpi_i}$. Thus we obtain

**Theorem 2.1.6.** For $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, we have

\[
\mathbb{B}(m\varpi_i) \cong \text{Par}_{<m} \otimes \mathbb{B}_0(m\varpi_i) \text{ as crystals.}
\]

On the other hand, we know the following theorem from [BN, Theorem 4.16 (i)].
Theorem 2.1.7. For each $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, we have

\[ B(m \varpi_i) \cong \text{Par}_{<m} \otimes B_0(m \varpi_i) \] as crystals.

By combining Theorems 2.1.6 and 2.1.7 with Corollary 2.1.4, we can get our isomorphism theorem (Theorem 2.1.1).

Now, for an integral weight $\lambda \in P$, we set

\[ B(\lambda)_{cl} := \{ \text{cl}(\pi) \mid \pi \in B(\lambda) \} , \]

where for a path $\pi$, we define $\text{cl}(\pi) : [0, 1] \to \mathbb{Q} \otimes \mathbb{Z} P_{cl} \cong \mathfrak{h}^*/\mathbb{Q}\delta$ by: $(\text{cl}(\pi))(t) := \text{cl}(\pi(t))$ for $t \in [0, 1]$. We can endow $B(\lambda)_{cl}$ with a structure of crystal with weight lattice $P_{cl}$ in such a way that

\[
\begin{align*}
    e_j \text{cl}(\pi) &:= \text{cl}(e_j \pi), & f_j \text{cl}(\pi) &:= \text{cl}(f_j \pi), \\
    e_j(\text{cl}(\pi)) &:= e_j(\pi), & \varphi_j(\text{cl}(\pi)) &:= \varphi_j(\pi), \\
    \text{wt}(\text{cl}(\pi)) &:= \text{cl}(\text{wt}(\pi)).
\end{align*}
\]

for $\pi \in B(\lambda)$ and $j \in I$ (see [NS2, §3.3] and [NS3, §§1.3 and 1.4]). The following is a consequence of Theorem 2.1.1 (see [NS1, Proposition 5.8] and [NS2, Proposition 3.2]).

Theorem 2.1.8. For each $i \in I_0$, the crystal $B(\varpi_i)_{cl}$ is isomorphic to the crystal base $B(\varpi_i)_{cl}$ of the fundamental module $W(\varpi_i)$ of level zero as a crystal with weight lattice $P_{cl}$.

2.2 Tensor product decomposition theorem. In [NS3], we studied the crystal structure of $B(\lambda)_{cl} = \text{cl}(B(\lambda))$ for a general integral weight $\lambda \in P$ of level zero. Before stating our main result in [NS3], we make some comments. Let $\lambda \in P$ be an integral weight of level zero. We can write the $\lambda \in P$ in the form

\[ \lambda = \sum_{i \in I_0} m'_i \varpi_i + R\delta \]

for some $m'_i \in \mathbb{Z}$, $i \in I_0$, and $R \in \mathbb{Q}$ (cf. [Kac, Chap. 6]). Then it follows from the definition of LS paths that

\[ B(\lambda) = \{ \pi + \pi_{R\delta} \mid \pi \in B(\sum_{i \in I_0} m'_i \varpi_i) \} , \]

where we set $(\pi + \pi_{R\delta})(t) := \pi(t) + tR\delta$, $t \in [0, 1]$, and from the definition of the root operators that the crystal graph of $B(\lambda)$ is the same shape as that
of $\mathbb{B}(\sum_{i \in I_0} m'_i \varpi_i)$, up to $R$-shift of weight. Therefore we have that $\mathbb{B}(\lambda)_{c1} = \mathbb{B}(\sum_{i \in I_0} m'_i \varpi_i)_{c1}$. In addition, the integral weight $\sum_{i \in I_0} m'_i \varpi_i \in P$ is equivalent to the one that is dominant with respect to the simple coroots $\{h'_j\}_{j \in I_0}$ under the Weyl group $\hat{W} := \langle r_j \mid j \in I_0 \rangle \subset W$ (of finite type). Hence there exist nonnegative integers $m_i \in \mathbb{Z}_{\geq 0}, i \in I_0$, such that $\mathbb{B}(\sum_{i \in I_0} m'_i \varpi_i)_{c1}$. In addition, the integral weight $\sum_{\iota \in I_0} m'_i \varpi_i \in P$ is equivalent to the one that is dominant with respect to the simple coroots $\{h'_j\}_{j \in I_0}$ under the Weyl group $W^0 := \langle r_j \mid j \in I_0 \rangle \subset W$ (of finite type). Hence there exist nonnegative integers $m_i \in \mathbb{Z}_{\geq 0}, i \in I_0$, such that $\mathbb{B}(\sum_{i \in I_0} m_i \varpi_i)_{c1}$. Therefore we have that $\mathbb{B}(\lambda)_{c1} = \mathbb{B}(\sum_{i \in I_0} m_i \varpi_i)_{c1}$. To sum up, an integral weight $\lambda \in P$ of level zero, there exists $m_i \in \mathbb{Z}_{\geq 0}, i \in I_0$, such that $\mathbb{B}(\lambda)_{c1} = \mathbb{B}(\sum_{i \in I_0} m_i \varpi_i)_{c1}$. Thus, when we study the crystal $\mathbb{B}(\lambda)_{c1}$ for an integral weight $\lambda \in P$ of level zero, we may assume that the $\lambda \in P$ is of the form: $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$ from the beginning.

Now we are ready to state our main result in [NS3].

**Theorem 2.2.1 ([NS3, Theorem 2.2.1]).** Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. Then, there exists an isomorphism $\mathbb{B}(\lambda)_{c1} \simotimes \bigotimes_{i \in I_0} (\mathbb{B}(\varpi_i)_{c1})^{\otimes m_i}$ of crystals (with weight lattice $P_{c1}$) between $\mathbb{B}(\lambda)_{c1}$ and the tensor product $\bigotimes_{i \in I_0} (\mathbb{B}(\varpi_i)_{c1})^{\otimes m_i}$ of the crystals $\mathbb{B}(\varpi_i)_{c1}, i \in I_0$.

By combining Theorems 2.1.8 and 2.2.1, we obtain the next corollary.

**Corollary 2.2.2.** Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. The crystal $\mathbb{B}(\lambda)_{c1}$ is, as a crystal (with weight lattice $P_{c1}$), isomorphic to the crystal base $\bigotimes_{i \in I_0} (\mathbb{B}(\varpi_i)_{c1})^{\otimes m_i}$ of the tensor product $\bigotimes_{i \in I_0} W(\varpi_i)^{\otimes m_i}$ of fundamental $U'_q(\mathfrak{g})$-modules $W(\varpi_i), i \in I_0$, of level zero.

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**References**


