Perfectness and Multicoloring of Unit Disk Graphs on Triangular Lattice Points

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概要

Given a pair of non-negative integers $m$ and $n$, $P(m, n)$ denotes a subset of 2-dimensional triangular lattice points defined by $P(m, n) \overset{\text{def}}{=} \{(xe_1 + ye_2) \mid x \in \{0, 1, \ldots, m-1\}, y \in \{0,1,\ldots,n-1\}\}$ where $e_1 \overset{\text{def}}{=} (1,0), e_2 \overset{\text{def}}{=} (1/2,\sqrt{3}/2)$. Let $T_{m,n}(d)$ be an undirected graph defined on vertex set $P(m, n)$ satisfying that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to $d$. In this paper, we discuss a necessary and sufficient condition that $T_{m,n}(d)$ is perfect. More precisely, we show that $\forall m \in \mathbb{Z}_+, T_{m,n}(d)$ is perfect if and only if $d \geq \sqrt{n^2-3n+3}$.

Given a non-negative vertex weight vector $\mathbf{w} \in \mathbb{Z}_+^{P(m,n)}$, a multicoloring of $(T_{m,n}(d), \mathbf{w})$ is an assignment of colors to $P(m, n)$ such that each vertex $v \in P(m, n)$ admits $\mathbf{w}(v)$ colors and every adjacent pair of two vertices does not share a common color. We also give an efficient algorithm for multicoloring $(T_{m,n}(d), \mathbf{w})$ when $P(m, n)$ is perfect.

In general case, our results on the perfectness of $P(m, n)$ implies a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d), \mathbf{w})$. Our algorithm finds a multicoloring which uses at most $\alpha(d)\omega + O(d^3)$ colors, where $\omega$ denotes the weighted clique number. When $d = 1, \sqrt{3}/2, \sqrt{7}/3$, the approximation ratio $\alpha(d) = (4/3), (5/3), (5/3), (7/4), (7/4)$, respectively. When $d > 1$, we showed that $\alpha(d) \leq \left(1 + \frac{2}{\sqrt{3+\frac{2}{d^2-1}}}\right)$.

We also showed the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d), \mathbf{w})$ with strictly less than $(4/3)\omega$ colors.

1 Introduction

Given a pair of non-negative integers $m$ and $n$, $P(m, n)$ denotes the subset of 2-dimensional integer triangular lattice points defined by

$P(m, n) \overset{\text{def}}{=} \{(xe_1 + ye_2) \mid x \in \{0, 1, 2, \ldots, m-1\}, y \in \{0,1,2,\ldots,n-1\}\}$

where $e_1 \overset{\text{def}}{=} (1, 0), e_2 \overset{\text{def}}{=} (1/2, \sqrt{3}/2)$. Given a finite set of 2-dimensional points $P \subseteq \mathbb{R}^2$ and a positive real $d$, a unit disk graph, denoted by $(P,d)$, is an undirected graph with vertex set
$P$ such that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to $d$. We denote the unit disk graph $(P(m, n), d)$ by $T_{m,n}(d)$.

Given an undirected graph $H$ and a non-negative integer vertex weight $w'$ of $H$, a multicoloring of $(H, w')$ is an assignment of colors to vertices of $H$ such that each vertex $v$ admits $w'(v)$ colors and every adjacent pair of two vertices does not share a common color. A multicoloring problem on $(H, w')$ finds a multicoloring of $(H, w')$ which minimizes the required number of colors. The multicoloring problem is also known as weighted coloring [4], minimum integer weighted coloring [15] or $w$-coloring [12].

In this paper, we study weighted unit disk graphs on triangular lattice points $(T_{m,n}(d), w)$. First, we show a necessary and sufficient condition that $T_{m,n}(d)$ is a perfect graph. If the graph is perfect, we can solve the multicoloring problem easily. Next, we propose a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d), w)$. Our algorithm is based on the well-solvable case that the given graph is perfect. For any $d \geq 1$, our algorithm finds a multicoloring which uses at most

\[
1 + \left(1 + \frac{\lfloor \frac{2}{\sqrt{3}} d \rfloor}{\lfloor \frac{3 + \sqrt{4d^2 - 3}}{2} \rfloor} \right) \omega + \left(\lfloor \frac{3 + \sqrt{4d^2 - 3}}{2} \rfloor - 1\right) \lfloor d + 1 \rfloor^2
\]

colors, where $\omega$ denotes the weighted clique number. Table 1 shows the values of the above approximation ratio in case that $d$ is small.

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<tr>
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We also show the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d), w)$ which uses strictly less than $(4/3)\omega$ colors.

The multicoloring problem has been studied in several context. When a given graph is the triangular lattice graph $T_{m,n}(1)$, the problem is related to the radio channel (frequency) assignment problem. McDiarmid and Reed [9] showed that the multicoloring problem on triangular lattice graphs is NP-hard. Some authors [9, 12] independently gave $(4/3)$-approximation algorithms for this problem. In case that a given graph $H$ is a square lattice graph or a hexagonal lattice graph, the graph $H$ becomes bipartite and so we can obtain an optimal multicoloring of $(H, w')$ in polynomial time (see [9] for example). Halldórsson and Kortsarz [5] studied planar graphs and partial $k$-trees. For both classes, they gave a polynomial time approximation scheme (PTAS) for variations of multicoloring problem with min-sum objectives. These objectives appear in the context of multiprocessor task scheduling. For coloring (general) unit disk graphs, there exists a 3-approximation algorithm [6, 8, 14]. Here we note that the approximation ratio of our algorithm is less than $1 + 2/\sqrt{3} < 2.155$ for any $d \geq 1$. 
2 Well-Solvable Cases and Perfectness

In this section, we discuss some well-solvable cases such that the multicoloring number is equivalent to the weighted clique number.

An undirected graph $G$ is perfect if for each induced subgraph $H$ of $G$, the chromatic number of $H$, denoted by $\chi(H)$, is equal to its clique number $\omega(H)$. The following theorem is a main result of this paper.

**Theorem 1** When $n \geq 1$ and $d \geq 1$, we have the following:

\[
[\forall m \in \mathbb{Z}_+, T_{m,n}(d) \text{ is perfect } \text{ if and only if } d \geq \sqrt{n^2 - 3n + 3}.
\]

Table 2 shows the perfectness and imperfectness of $T_{m,n}(d)$ for small $n$ and $d$.

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To show the above theorem, we introduce some definitions. We say that an undirected graph has a transitive orientation property, if each edge can be assigned a one-way direction in such a way that the resulting directed graph $(V, F)$ satisfies that $[(a, b) \in F$ and $(b, c) \in F$ imply $(a, c) \in F]$. An undirected graph which is transitively orientable is called comparability graph. The complement of a comparability graph is called co-comparability graph. It is well-known that every co-comparability graph is perfect.

**Lemma 1** For any integer $n \geq 1$, if $d \geq \sqrt{n^2 - 3n + 3}$, then $T_{m,n}(d)$ is a co-comparability graph.

**Proof:** omitted.

The following lemma deals with the special case that $n = 3$, $d = 1$.

**Lemma 2** For any $m \in \mathbb{Z}_+$ and $1 \leq d < \sqrt{3}$, the graph $T_{m,3}(d)$ is perfect.

**Proof:** We only need to consider the case that $d = 1$, since $T_{m,n}(d) = T_{m,n}(1)$ when $1 \leq d < \sqrt{3}$. Let $H$ be an induced subgraph of $T_{m,3}(1)$. When $\omega(H) \leq 2$, $H$ has no 3-cycle. Then it is easy to show that $H$ has no odd cycle and thus $\chi(H) = \omega(H)$, since $H$ is bipartite. If $\omega(H) \geq 3$, then it is clear that $\omega(H) = 3$ and $\chi(H) \leq 3$, since $\omega(T_{m,3}(1)) = 3$ and $T_{m,3}(1)$ has a trivial 3-coloring.

Note that though the graph $T_{m,3}(1)$ is perfect, the graph $T_{m,3}(1)$ is not co-comparability graph.
From the above, the perfectness of a graph satisfying the conditions of Theorem 1 is clear. In the following, we discuss the inverse implication. We say that an undirected graph $G$ has an odd-hole, if $G$ contains an induced subgraph isomorphic to an odd cycle whose length is greater than or equal to 5. It is obvious that if a graph has an odd-hole, the graph is not perfect. In the following, we denote a point $(x_1 + y_2) \in P(m, n)$ by $(x, y)$.

**Lemma 3** If $1 \leq d < \sqrt{3}$, then $\forall m \geq 5$, $T_{m,4}(d)$ has at least one odd-hole.

**Proof:** If $1 \leq d < \sqrt{3}$, then a subgraph induced by \{ $\langle 2, 0 \rangle$, $\langle 1, 1 \rangle$, $\langle 0, 2 \rangle$, $\langle 0, 3 \rangle$, $\langle 1, 3 \rangle$, $\langle 2, 3 \rangle$, $\langle 3, 2 \rangle$, $\langle 3, 1 \rangle$, $\langle 3, 0 \rangle$ \} is a 9-hole. If $\sqrt{3} \leq d < 2$, then a subgraph induced by \{ $\langle 3, 0 \rangle$, $\langle 1, 1 \rangle$, $\langle 0, 2 \rangle$, $\langle 1, 3 \rangle$, $\langle 2, 3 \rangle$, $\langle 4, 2 \rangle$, $\langle 4, 1 \rangle$ \} is a 7-hole. If $2 \leq d < \sqrt{7}$, then a subgraph induced by \{ $\langle 2, 0 \rangle$, $\langle 0, 2 \rangle$, $\langle 1, 3 \rangle$, $\langle 3, 2 \rangle$, $\langle 3, 0 \rangle$ \} is a 5-hole. When $1 \leq d < \sqrt{7}$, $T_{5,4}(d)$ has at least one odd-hole, and hence the proof is completed.

**Lemma 4** If $1 \leq d < \sqrt{13}$, then $\forall m \geq 6$, $T_{m,5}(d)$ has at least one odd-hole.

**Proof:** If $1 \leq d < \sqrt{13}$, then odd-holes in the proof of Lemma 3 are induced subgraph of $T_{5,5}(d)$. If $\sqrt{7} \leq d < 3$, then a subgraph induced by \{ $\langle 2, 0 \rangle$, $\langle 0, 2 \rangle$, $\langle 1, 4 \rangle$, $\langle 4, 2 \rangle$, $\langle 4, 0 \rangle$ \} is a 5-hole. If $3 \leq d < \sqrt{13}$, then a subgraph induced by \{ $\langle 3, 0 \rangle$, $\langle 0, 3 \rangle$, $\langle 2, 4 \rangle$, $\langle 5, 3 \rangle$, $\langle 5, 0 \rangle$ \} is a 5-hole. When $1 \leq d < \sqrt{13}$, $T_{6,5}(d)$ has at least one odd-hole, and hence the proof is completed.

**Lemma 5** For any integer $n \geq 4$, if $1 \leq d < \sqrt{n^2 - 3n + 3}$, then $\exists m \in \mathbb{Z}_+$, $T_{m,n}(d)$ is imperfect.

**Proof:** In the following, we show that $\forall n \geq 4$, if $1 \leq d < \sqrt{n^2 - 3n + 3}$, then $\exists m \in \mathbb{Z}_+$, $T_{m,n}(d)$ has at least one odd-hole, by induction on $n$. When $n = 4, 5$, it is clear from Lemmas 3 and 4, respectively.

Now we consider the case that $n = n' \geq 6$ under the assumption that if $1 \leq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3}$, then $\exists m' \in \mathbb{Z}_+$, $T_{m',n'-1}(d)$ has at least one odd-hole. If $1 \leq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3} = \sqrt{n'^2 - 5n' + 7}$, then $T_{m',n'}(d)$ has at least one odd-hole, since $T_{m',n'-1}(d)$ is an induced subgraph of $T_{m',n'}(d)$. In the remained case that $\sqrt{n'^2 - 5n' + 7} \leq d < \sqrt{n'^2 - 3n' + 3}$, the set of points \{ $\langle n' - 3, 0 \rangle$, $\langle 0, n' - 2 \rangle$, $\langle n' - 4, n' - 1 \rangle$, $\langle 2n' - 7, n' - 2 \rangle$, $\langle 2n' - 6, 0 \rangle$ \} is contained in $P(m'', n')$, if $m'' = 2n' - 5$. It is easy to see that the above five vertices induces a 5-hole of $T_{m'',n'}(d)$, when $n' \geq 6$ and $\sqrt{n'^2 - 5n' + 7} \leq d < \sqrt{n'^2 - 3n' + 3}$

Lemma 5 shows the imperfectness of every graph which violates a condition of Theorem 1. Thus, we completed a proof of Theorem 1. From the above lemmas, the following is immediate.

**Corollary 1** Let $d > 1$ be a real number. Then, $T_{m,n}(d)$ is a co-comparability graph, if and only if $n \leq \frac{3 + \sqrt{13}}{2}$. 

Lastly, we discuss some algorithmic aspects. Assume that we have a co-comparability graph $G$ and related digraph $H$ which gives a transitive orientation of the complement of $G$. Then each independent set of $G$ corresponds to a chain (directed path) of $H$. The multicoloring
problem on $G$ is essentially equivalent to the minimum size chain cover problem on $H$. Every clique of $G$ corresponds to an anti-chain of $H$. Thus the equality $\omega(G) = \chi(G)$ is obtained from Dilworth’s decomposition theorem [2]. It is well-known that the minimum size chain cover problem on an acyclic graph is solvable in polynomial time by using an algorithm for minimum-cost circulation flow problem (see [13] for example).

Though an weighted graph $(T_{m,n}, w)$ is not a co-comparability graph, we can construct exact multicoloring algorithm for the graph. Here we omit the detail.

## 3 Approximation Algorithm

In this section, we propose an approximation algorithm for multicoloring the graph $(T_{m,n}(d), w)$. When $d = 1$, McDiarmid and Reed [9] proposed an approximation algorithm for $(T_{m,n}(1), w)$, which finds a multicoloring with at most $(4/3)\omega(T_{m,n}(1), w) + 1/3$ colors.

In the following, we propose an approximation algorithm for $(T_{m,n}(d), w)$ when $d > 1$. The basic idea of our algorithm is similar to the shifting strategy [7].

**Theorem 2** When $d > 1$, there exists a polynomial time algorithm for multicoloring $(T_{m,n}(d), w)$ such that the number of required colors is bounded by

$$
1 + \left(1 + \frac{\frac{2\sqrt{3}d}{3+\sqrt{4d^2-3}}}{3+\sqrt{4d^2-3}}\right)\omega(T_{m,n}(d), w) + \left(1 + \frac{\frac{2\sqrt{3}d}{3+\sqrt{4d^2-3}}}{3+\sqrt{4d^2-3}}\right)\chi(T_{m,n}(d)).
$$

**Proof:** We describe an outline of the algorithm. For simplicity, we define $K_1 = \left\lceil \frac{3 + \sqrt{4d^2 - 3}}{2} \right\rceil$, and $K_2 = \left\lceil \frac{3 + \sqrt{4d^2 - 3}}{2} \right\rceil + \left\lceil \frac{\sqrt{3}}{2}d \right\rceil$.

First, we construct $K_2$ vertex weights $w'_k$ for $k \in \{0, 1, \ldots, K_2 - 1\}$ by setting

$$
w'_k(x, y) = \begin{cases} 0, & y \in \{k, k+1, \ldots, k + \left\lfloor \frac{2\sqrt{3}d}{\sqrt{3}} \right\rceil\right\rceil, \text{mod } K_2, \\ \left\lfloor \frac{w(x, y)}{K_1} \right\rceil, & \text{otherwise.} \end{cases}
$$

Next, we exactly solve $K_2$ multicoloring problems defined by $K_2$ pairs $(T_{m,n}(d), w'_k)$, $k \in \{0, 1, \ldots, K_2 - 1\}$ and obtain $K_2$ multicolorings. We can solve each problem exactly in polynomial time, since every connected component of the graph induced by the set of vertices with positive weight is a perfect graph discussed in the previous section. Thus $\chi(T_{m,n}(d), w'_k) = \omega(T_{m,n}(d), w'_k)$ for any $k \in \{0, 1, \ldots, K_2 - 1\}$. Put $w'' = w - \sum_{k=0}^{K_2-1} w'_k$. Then each element of $w''$ is less than or equal to $K_1 - 1$. Thus we can find a multicoloring of $(T_{m,n}(d), w'')$ from the direct sum of $K_1 - 1$ trivial colorings of $T_{m,n}(d)$. The obtained multicoloring uses at most $(K_1 - 1)\chi(T_{m,n}(d))$ colors. Lastly, we output the direct sum of $K_2 + 1$ multicolorings obtained above. The definition of the weight vector $w'_k$ implies that $\forall k \in \{0, 1, \ldots, K_2 - 1\}$, $K_1 \omega(T_{m,n}(d), w'_k) \leq \omega(T_{m,n}(d), w)$. Thus, the obtained multicoloring uses at most $(K_2/K_1)\omega(T_{m,n}(d), w) + (K_1 - 1)\chi(T_{m,n}(d))$ colors.

The following lemma gives the chromatic number of $T_{m,n}(d)$.
Lemma 6 If $m, n$ are sufficiently large, then $\chi(T_{m,n}(d)) = \hat{d}^2$ where $\hat{d}$ is the minimum Euclidean distance between two points in $P(m,n)$ subject to that distance being greater than $d$. Clearly, $d < \hat{d} \leq \lfloor d + 1 \rfloor$.

Proof: See McDiarmid [9] for example.

When $d$ is small, Table 1 shows the approximation ratio. The following corollary gives a simple upper bound of the approximation ratio.

Corollary 2 For any $d \geq 1$, we have $1 + \frac{\frac{\sqrt{3}d}{3 + \sqrt{4d - 3}}}{2} \leq 1 + \frac{2}{\sqrt{3} + 2\sqrt{3} - 3}$.

Here we note that if we apply our algorithm in the case that $d = 1$, then the algorithm finds a multicoloring which uses at most $(4/3)\omega(T_{m,n}(1), w) + 6$ colors.

4 Discussion

In this paper, we dealt with the triangular lattice. In the following, we discuss the square lattice. Given a pair of non-negative integers $m$ and $n$, $Q(m,n) \stackrel{\text{def}}{=} \{0, 1, 2, \ldots, m - 1\} \times \{0, 1, 2, \ldots, n - 1\}$ denotes the subset of 2-dimensional integer square lattice points. We denote the unit disk graph $(Q(m,n), d)$ by $S_{m,n}(d)$. In case that $d < \sqrt{2}$, it is clear that $S_{m,n}(d) = S_{m,n}(1)$ and the graph is bipartite for any $m$ and $n$. If $d = \sqrt{2}$, we proposed a $(4/3)$-approximation algorithm for multicoloring $(S_{m,n}(\sqrt{2}), w)$ in our previous paper [11]. We also showed the NP-hardness of the problem.

Unfortunately, Theorem 1 is not extensible to the square lattice case. Table 3 shows the perfectness and imperfectness of unit disk graphs on the square lattice for small $n$ and $d$. The perfectness of $T_{m,3}(\sqrt{2})$ was shown in [11]. The graph $S_{m,3}(2)$ contains a 5-hole: $\{(0,0), (2,0), (2,1), (1,2), (0,2)\}$.

表 3: Unit disk graphs on square lattice points

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