Jørgensen groups of parabolic type II

(Countable infinite case)

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ABSTRACT. This paper is the second part of the study on Jørgensen groups of parabolic type. We will show that there are countable infinite many Jørgensen groups of parabolic type on a certain cylinder in this case.

1. Introduction.

1.1. It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup $G$ of the Möbius transformation group is discrete. For this problem there are two important and useful theorems: One is Poincaré's polyhedron theorem, which is a sufficient condition for $G$ to be discrete. The other is Jørgensen's inequality, which is a necessary condition for a two-generator Möbius transformation group $\langle A, B \rangle$ to be discrete. Here we will consider extreme discrete groups (Jørgensen groups) for Jørgensen's inequality. This paper is the second part

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of a series of studies on Jørgensen groups (cf. Li - Oichi - Sato [4]).

1.2. Let Möb denote the set of all linear fractional transformations (Möbius transformations)

\[ A(z) = \frac{az + b}{cz + d} \]

of the extended complex plane \( \hat{C} = \mathbb{C} \cup \{\infty\} \), where \( a, b, c, d \) are complex numbers and the determinant \( ad - bc = 1 \). There is an isomorphism between Möb and \( PSL(2, \mathbb{C}) \). We always write elements of Möb as matrices with determinant 1 in this paper. We recall that Möb (\( = PSL(2, \mathbb{C}) \)) acts on the upper half space \( H^3 \) of \( \mathbb{R}^3 \) as the group of conformal isometries of hyperbolic 3-space.

In this paper we use a Kleinian group in the same meaning as a discrete group. Namely, a Kleinian group is a discrete subgroup of Möb. A Kleinian group \( G \) is of the first kind if the limit set \( \Lambda(G) \) of \( G \) is all of the extended complex plane \( \hat{C} \) and it is of the second kind otherwise. A subgroup \( G \) of Möb is said to be elementary if there exists a finite \( G \)-orbit in \( \mathbb{R}^3 \).

1.3. The trace \( \text{tr}(A) \) of the matrix

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1) \]

in \( SL(2, \mathbb{C}) \) is defined by \( \text{tr}(A) = a + d \). We remark that the trace of an element of Möb (\( = PSL(2, \mathbb{C}) \)) is not well-defined, but Jørgensen number (defined later) is still well-defined after choosing matrix representatives.

1.4. Let \( A^* \) and \( B^* \) be matrices in \( SL(2, \mathbb{C}) \) representing the Möbius transformations \( A \) and \( B \), respectively. As \( A^* \) and \( B^* \) are determined by \( A \) and \( B \) to within a factor of \(-1\), we see that the commutator \( A^*B^*(A^*)^{-1}(B^*)^{-1} \) (resp. \( (A^*)^2 \)) are uniquely determined by \( A \) and \( B \) (resp. \( A \)). Thus we may write \( \text{tr}(ABA^{-1}B^{-1}) = \text{tr}(A^*B^*(A^*)^{-1}(B^*)^{-1}) \) and \( \text{tr}^2(A) = \text{tr}^2(A^*) \).
In 1976 Jørgensen obtained the following important theorem, which gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B \rangle$ to be discrete.

**Theorem A (Jørgensen [1]).** Suppose that the Möbius transformations $A$ and $B$ generate a non-elementary discrete group. Then

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$  

The lower bound 1 is best possible.

1.5. **Definition 1.** Let $A$ and $B$ be Möbius transformations. The Jørgensen number $J(A, B)$ for the ordered pair $(A, B)$ is defined by

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2|.$$  

**Definition 2.** A subgroup $G$ of Möb is called a Jørgensen group if $G$ satisfies the following four conditions:

1. $G$ is a two-generator group.
2. $G$ is a discrete group.
3. $G$ is a non-elementary group.
4. There exist generators $A$ and $B$ of $G$ such that $J(A, B) = 1$.

1.6 Jørgensen and Kiikka showed the following.

**Theorem B (Jørgensen-Kiikka [2]).** Let $\langle A, B \rangle$ be a non-elementary discrete group with $J(A, B) = 1$. Then $A$ is elliptic of order at least seven or $A$ is parabolic.

If $\langle A, B \rangle$ is a Jørgensen group such that $A$ is parabolic and $J(A, B) = 1$, then we call it a Jørgensen group of parabolic type. There are infinite many Jørgensen groups of parabolic type (Jørgensen-Lascurain-Pignataro [3], Sato [6]).

Now it gives rise to the following problem.
Problem 1. Find all Jörgensen groups of parabolic type.

1.7. Let \( \langle A, B \rangle \) be a marked two-generator group such that \( A \) is parabolic. Then we can normalize \( A \) and \( B \) as follows:
\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\sigma,\mu} = \begin{pmatrix} \mu \sigma & \mu^2 \sigma - 1/\sigma \\ \sigma & \mu \sigma \end{pmatrix}
\]
where \( \sigma \in \mathbb{C} \setminus \{0\} \) and \( \mu \in \mathbb{C} \). See [4] for this normalization.

We denote by \( G_{\sigma,\mu} \) the marked group generated by \( A \) and \( B_{\sigma,\mu} \): \( G_{\sigma,\mu} = \langle A, B_{\sigma,\mu} \rangle \). We say that \( (\sigma, \mu) \in \mathbb{C} \setminus \{0\} \times \mathbb{C} \) is the point representing a marked group \( G_{\sigma,\mu} \) and that \( G_{\sigma,\mu} \) is the marked group corresponding to a point \( (\sigma, \mu) \).

1.8. In [6], Sato considered the case of \( \mu = ik \) (\( k \in \mathbb{R} \)). Namely, he considered marked two-generator group \( G_{\sigma,ik} = \langle A, B_{\sigma,ik} \rangle \) generated by
\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\sigma,ik} = \begin{pmatrix} ik \sigma & -k^2 \sigma - 1/\sigma \\ \sigma & ik \sigma \end{pmatrix}
\]
where \( \sigma \in \mathbb{C} \setminus \{0\} \) and \( k \in \mathbb{R} \).

Now we have the following conjecture.

CONJECTURE. For any Jörgensen group \( G \) of parabolic type there exists a marked group \( G_{\sigma,ik} (\sigma \in \mathbb{C} \setminus \{0\}, k \in \mathbb{R}) \) such that \( G_{\sigma,ik} \) is conjugate to \( G \).

If this conjecture is true, then it is sufficient to consider the case of \( \mu = ik \) in order to find all Jörgensen groups of parabolic type. In this paper we only consider the case of \( \mu = ik \).

1.9. Let \( C \) be the following cylinder:
\[
C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbb{R} \}.
\]

Theorem C (Sato [6]). If a marked two-generator group \( G_{\sigma,ik}(\sigma \in \mathbb{C} \setminus \{0\}, k \in \mathbb{R}) \), then
(1) is a Jørgensen group, then the point $(\sigma, ik)$ representing $G_{\sigma, ik}$ lies on the cylinder $C$.

By Theorem C, if $(\theta, k)$ is a point on the cylinder $C$, then we can set $\sigma = -ie^{i\theta} (0 \leq \theta \leq 2\pi)$. If a point $(-ie^{i\theta}, ik)$ on the cylinder $C$ represents a Jørgensen group, then we say that the group is a Jørgensen group of parabolic type $(\theta, k)$.

Now it gives rise to the following problem.

**Problem 2.** Find all Jørgensen groups of parabolic type $(\theta, k)$.

We divide Jørgensen groups of this type into three parts as follows:

- **Part 1.** $|k| \leq \sqrt{3}/2$, $0 \leq \theta \leq 2\pi$ (finite case).
- **Part 2.** $\sqrt{3}/2 < |k| \leq 1$, $0 \leq \theta \leq 2\pi$ (countable infinite case).
- **Part 3.** $1 < |k|$, $0 \leq \theta \leq 2\pi$ (uncountable infinite case).

By some lemmas in [6], it suffices to consider the case of $0 \leq \theta \leq \pi/2$ and $k \geq 0$ in order to find Jørgensen groups of parabolic type $(\theta, k)$.

In the previous paper [4] we find all Jørgensen groups in the case where $0 \leq \theta \leq \pi/2$ and $0 \leq k \leq \sqrt{3}/2$, that is, we obtain the following theorem.

**Theorem D (finite case)** (Li - Oichi - Sato [4]).

(i) *There are sixteen Jørgensen groups in $D = \{(\theta, k) \in \mathbb{R} | 0 \leq \theta \leq \pi/2, 0 \leq k \leq \sqrt{3}/2 \}$.*

(ii) *Nine of them are Kleinian groups of the first kind and seven groups are of the second kind.*

1.10. For a sufficient condition for a subgroup of the Möbius transformation group to be discrete, the following theorem is well-known.

**Theorem E** (Poincaré’s Polyhedron Theorem (Maskit [5, p.73])).

Let $P$ be a polyhedron with side pairing transformations satisfying the following conditions (1) through (6). Then, $G$, the group generated by the side pairing trans-
formations, is discrete and $P$ is a fundamental polyhedron for $G$, and the reflection relations and cycle relations form a complete set of relations for $G$:

(1) For each side $s$ of $P$, there is a side $s'$ and there is an element $g_s \in G$ satisfying $g_s(s) = s'$ and $g_{s'} = g_s^{-1}$.

(2) $g_s(P) \cap P = \emptyset$.

(3) For every point $z \in P^*$, $p^{-1}(z)$ is a finite set. Here $P^*$ is the space of equivalence classes so that the projection $p : \bar{P}$ (the closure of $P$) $\to P^*$ is continuous and open.

(4) Let $e$ be an edge and let $h$ be the cycle transformation at $e$. Then for each edge $e$, there is a positive integer $t$ such that $h^t = 1$.

(5) Let $\{e_1, e_2, \ldots, e_m\}$ be any cycle of edges of $P$ and let $\alpha(e_k)$ $(k = 1, 2, \ldots, m)$ be the angle measured from inside $P$ at the edge $e_k$. Let $q$ be the smallest positive integer such that $h^q = 1$, where $h$ is the cycle transformation at $e_1$. Then the equality

$$\sum_{k=1}^{m} \alpha(e_k) = 2\pi/q$$

holds.

(6) $P^*$ is complete.

2. Theorems.

In this section we will state that we find all Jørgensen groups in Part 2, that is, we obtain the following theorems. The proofs will appear elsewhere.

Main Theorem. There are countable infinite many Jørgensen groups on the region $\{(\theta, k) | 0 \leq \theta \leq \pi/2, \sqrt{3}/2 < k < 1\}$.

For simplicity we write $B_{\theta,k}$ for $B_{-ie^{i\theta},ik}$.

This theorem consists of the following Theorem 1 through Theorem 6.
Let $A$ and $B_{\theta,k}$ ($k \in \mathbb{R}, 0 \leq \theta \leq \pi/2$) be the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & i e^{-i\theta} (k^2 e^{2i\theta} - 1) \\ -i e^{i\theta} & ke^{i\theta} \end{pmatrix}$$

We can prove these theorems by using Jørgensen's inequality and Poincaré's polyhedron theorem.

**Theorem 1** (Li - Oichi - Sato [4]). Let $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ be the group generated by $A$ and $B_{\theta,k}$. If $0 < \theta < \pi/6$, $\pi/6 < \theta < \pi/4$, $\pi/4 < \theta < \pi/3$, $\pi/3 < \theta < \pi/2$, then $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ are not Kleinian groups and so not Jørgensen groups for $k \in \mathbb{R}$.

**Theorem 2** (the case of $\theta = 0$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{0,k} = \begin{pmatrix} k & i(k^2 - 1) \\ -i & k \end{pmatrix} \quad (k \in \mathbb{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then the following hold.

(i) In the case where $\cos(\pi/2m) < k < \cos(\pi/(2m + 2))$ and $k \neq \cos(\pi/(2m + 1))$ $(m = 3, 4, \ldots)$, $G_k$ are not Kleinian groups and not Jørgensen groups.

(ii) In the case of $k = 1$, $G_k$ is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signature $(0; 2, 3, \infty)$.

(iii) In the case of $k = \cos(\pi/n)$ $(n = 7, 8, \ldots)$, $G_k$ is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signatures $(0; 2, 3, n)$ and $(0; 2, 3, \infty)$.

**Theorem 3** (the case of $\theta = \pi/6$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{\pi/6,k} = \begin{pmatrix} ke^{\pi i/6} & i(k^2 e^{\pi i/6} - e^{-\pi i/6}) \\ -i e^{\pi i/6} & ke^{\pi i/6} \end{pmatrix} \quad (k \in \mathbb{R}),$$
and let $G_k = \langle A, B_k \rangle$. Then $G_k$ are not Kleinian groups and not Jørgensen groups for $k$ with $\sqrt{3}/2 < k \leq 1$.

**Theorem 4** (the case of $\theta = \pi/4$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{k} := B_{\pi/4,k} = \begin{pmatrix} k e^{\pi i/4} & i(k^2 e^{\pi i/4} - e^{-\pi i/4}) \\ -ie^{\pi i/4} & k e^{\pi i/4} \end{pmatrix} \quad (k \in \mathbb{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then the following hold.

(i) In case of $\sqrt{3}/2 < k < 1$, $G_k$ are not Kleinian groups and not Jørgensen groups.

(ii) In the case of $k = 1$, $G_k$ is a Kleinian group of the first kind and a Jørgensen group. The volume $V(G_{\pi/4,1})$ of the 3-orbifold for $G_{\pi/4,1}$ is

$$V(G_{\pi/4,1}) = 8[2L(\pi/4) - L(\pi/12) - L(5\pi/12)],$$

where $L(\theta)$ is the Lobachevskii function:

$$L(\theta) = -\int_{0}^{\theta} \log|2\sin u| \, du.$$

**Theorem 5** (the case of $\theta = \pi/3$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{k} := B_{\pi/3,k} = \begin{pmatrix} k e^{\pi i/3} & i(k^2 e^{\pi i/3} - e^{-\pi i/3}) \\ -ie^{\pi i/3} & k e^{\pi i/3} \end{pmatrix} \quad (k \in \mathbb{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then $G_k$ are not Kleinian groups and not Jørgensen groups for $k$ with $\sqrt{3}/2 < k \leq 1$.

**Theorem 6** (the case of $\theta = \pi/2$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{k} := B_{\pi/2,k} = \begin{pmatrix} ik & -(k^2 + 1) \\ 1 & ik \end{pmatrix} \quad (k \in \mathbb{R}),$$
and let $G_k = \langle A, B_k \rangle$. Then the following hold.

(i) In the case where $\cos(\pi/(2n-1)) < k < \cos(\pi/(2n + 1))$ and $k \neq \cos(\pi/2n)$ ($n = 3, 4, \ldots$), $G_k$ are not Kleinian groups and not Jørgensen groups.

(ii) In the case of $k = 1$, $G_k$ is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a Riemann surface with signature $(0; 3, 3, \infty)$.

(iii) In the case of $k = \cos(\pi/n)$ ($n = 7, 8, \ldots$), $G_k$ are Kleinian groups of the second kind and Jørgensen groups, and $\Omega(G_k)/G_k$ is a Riemann surface with signature $(0; 3, 3, n)$.

References


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