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Kyoto University
HEIGHT FUNCTIONS OVER FUNCTION FIELDS

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For details of this talk, see [1], [2], [3] and [4].

1. Function fields

First of all, we fix two kinds of functions fields, namely, an arithmetic
function field and a geometric function field.
- An arithmetic function field is a finitely generated extension field of \( \mathbb{Q} \).
- A geometric function field is a finitely generated extension field of an
algebraically closed field.

2. Height function on \( \mathbb{P}^1(\mathbb{Q}) \)

First, let us review a height of a rational number. Roughly speaking, it
measures the complexity of rational numbers, and you may agree with the
following:

The complexity of rational numbers \( \Rightarrow \)
The magnitude of numerators and denominators

Hence, for \( a/b \in \mathbb{Q} \) (\( a, b \in \mathbb{Z} \) and \( a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z} \)), the complexity \( h \) of \( a/b \)
should be

\[
h = \log \max\{|a|, |b|\}.
\]

This gives rise to a height function \( h^{\text{arith}} \) on

\[
\mathbb{P}^1(\mathbb{Q}) = \{(a : b) \mid a, b \in \mathbb{Q}, (a, b) \neq (0, 0)\},
\]

namely, for \( x = (a : b) \) with \( a, b \in \mathbb{Z} \) and \( a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z} \),

\[
h^{\text{arith}}(x) = \log \max\{|a|, |b|\}.
\]

3. Height function on \( \mathbb{P}^1(\mathbb{Q}(t)) \)

For this purpose, we need to ask again what

the complexity of polynomials
3.1. Geometric case. (Complexity = Degree)
For $x = (f(t) : g(t))$ with $f(t), g(t) \in \mathbb{Z}[t]$ and $f(t), g(t)$ relatively prime,

$$h_{\text{geom}}(x) = \max\{\deg(f(t)), \deg(g(t))\}.$$ 

$h_{\text{geom}}$ is NOT an extension of $h_{\text{arith}}$ when we view $\mathbb{Q}$ as a subfield of $\mathbb{Q}(t)$.

3.2. Arithmetic case. (Complexity = Degree + Largeness of coefficients)
For $f = \sum a_i t^i \in \mathbb{Q}[t]$, we set

$$|f|_\infty = \max_i |a_i|.$$ 

Then, as before, we may consider

$$\max\{\deg(f(t)), \deg(g(t))\} + \log \max\{|f|_\infty, |g|_\infty\},$$

which is NOT good from the geometric view point. Thus, we need a more sophisticated invariant to measure the largeness of coefficients. For this purpose, let us fix a positive $(1, 1)$-form $\Omega$ on $\mathbb{P}^1(\mathbb{C})$ with $\int_{\mathbb{P}^1(\mathbb{C})} \Omega = 1$. Then, we set

$$v(f) = \exp\left(\int_{\mathbb{P}^1(\mathbb{C})} \log |f|\Omega\right).$$

We can see $||f||_\infty = v(f)$. Hence, we may define

$$h_{\text{arith}}(x) = \max\{\deg(f(t)), \deg(g(t))\} + \int_{\mathbb{P}^1(\mathbb{C})} \log \max\{|f(t)|, |g(t)|\}\Omega.$$ 

4. A QUICK REVIEW OF ARAKELOV GEOMETRY

4.1. Arithmetic curve. Let $K$ be a number field and $O_K$ the ring of integers in $K$. Let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$. Let $L$ be a flat and finitely generated $O_K$-module of rank 1. For an embedding $\sigma \in K(\mathbb{C})$, the tensor product $L \otimes_K \mathbb{C}$ in terms of the embedding $\sigma$ is denoted by $L \otimes_\sigma \mathbb{C}$. The collection $(L, \{\| \cdot \|_\sigma\}_{\sigma \in K(\mathbb{C})})$ is called a hermitian line bundle on $C = \text{Spec}(O_K)$. For simplicity, it is denoted by $\overline{L}$.

Let $s$ be a non-zero element of $L$. Then, let us consider:

$$\log \#(L/sL) - \sum_\sigma \log(\|s \otimes_\sigma 1\|_\sigma).$$

Then, by the product formula, it does not depend on the choice of $s$, so that it is denoted by $\overline{\deg}(L)$. 

4.2. General case.

$X$: a projective and flat integral scheme over $\mathbb{Z}$ such that $X \to \text{Spec}(\mathbb{Z})$ is smooth over $\mathbb{Q}$.

$(Z, T)$: for a non-negative integer $p$, a pair $(Z, T)$ is called an arithmetic cycle codimension $p$ if $Z$ is a cycle of codimension $p$ and $T$ is a current of type $(p-1, p-1)$ on $X(\mathbb{C})$.

$\hat{Z}^p(X)$: the set of all arithmetic cycles of codimension $p$.

$\hat{R}^p(X)$: the subgroup of $\hat{Z}^p(X)$ generated by the following elements:

1. $((/), -\lfloor \log|f|^2 \rfloor)$, where $f$ is a non-zero rational function on an integral closed subscheme $Y$ of codimension $p-1$ and $\lfloor \log|f|^2 \rfloor$ is the current defined by

$$\lfloor \log|f|^2 \rfloor(\gamma) = \int_{Y(\mathbb{C})} (\log|f|^2) \gamma.$$

2. $(0, \partial(\alpha) + \bar{\partial}(\beta))$, where $\alpha$ and $\beta$ are currents of type $(p-2, p-1)$ and $(p-1, p-2)$ respectively.

Note that $\hat{Z}^0(X) = \mathbb{Z}(X, 0)$ and $\hat{R}^0(X) = 0$.

Here we define

$$\overline{\text{CH}}^p(X) := \hat{Z}^p(X)/\hat{R}^p(X).$$

Let $\overline{L} = (L, || \cdot ||)$ be a $C^\infty$-hermitian line bundle on $X$, that is, $L$ is a line bundle on $X$ and $|| \cdot ||$ is a $C^\infty$-hermitian metric of $L_{\mathbb{C}}$ on $X(\mathbb{C})$. We define a homomorphism

$$\hat{c}_1(\overline{L}) : \overline{\text{CH}}^p(X) \to \overline{\text{CH}}^{p+1}(X)$$

in the following way: Let $(Z, T)$ be an element of $\hat{Z}^p(X)$. For simplicity, we assume that $Z$ is integral. Then, taking a non-zero rational section $s$ of $L|_Z$, we consider an arithmetic cycle of codimension $p+1$:

$$(\text{div}(s) \text{ on } Z, -\lfloor \log\|s\|^2_Z \rfloor + c_1(\overline{L}) \wedge T),$$

where $\lfloor \log\|s\|^2_Z \rfloor$ is a current given by $\phi \mapsto \int_{Z(\mathbb{C})} \log\|s\|^2_Z \phi$.

Let $\overline{L}_1, \ldots, \overline{L}_{\dim X}$ be $C^\infty$-hermitian line bundles on $X$. Then,

$$\hat{c}_1(\overline{L}_1) \cdots \hat{c}_1(\overline{L}_{\dim X}) \in \overline{\text{CH}}^{\dim X}(X).$$

Moreover, we have a homomorphism

$$\overline{\text{deg}} : \overline{\text{CH}}^{\dim X}(X) \to \mathbb{R}$$

given by

$$\overline{\text{deg}} \left( \sum_P n_P P, T \right) = \sum_P n_P \log \#(\kappa(P)) + \frac{1}{2} \int_{X(\mathbb{C})} T.$$
Thus, we have the number
\[ \hat{\deg}(\hat{c}_1(L_1) \cdots \hat{c}_1(L_{\dim X})) , \]
which is called the intersection number of \( L_1, \ldots, L_{\dim X} \). Note that the intersection number
\[ \hat{\deg}(\hat{c}_1(L_1) \cdots \hat{c}_1(L_{\dim X})) \]
can be defined even if \( X \to \text{Spec}(\mathbb{Z}) \) is not smooth over \( \mathbb{Q} \).

5. Polarization and Height Function

**\( K \)**: an arithmetic function field, i.e., a field finitely generated over \( \mathbb{Q} \).

**\( d \)**: the transcendental degree of \( K \) over \( \mathbb{Q} \).

**\( B \)**: a projective and flat integral scheme over \( \mathbb{Z} \) whose function field is \( K \).

**\( \overline{H} \)**: a nef hermitian line bundle on \( B \), i.e., the Chern form \( c_1(\overline{H}) \) on \( B(\mathbb{C}) \) is semi-positive and \( \hat{\deg} (\hat{c}_1(\overline{H}) \cdot (Z, 0)) \geq 0 \) for every integral 1-dimensional subscheme \( Z \) on \( B \).

\((B, \overline{H})\): A pair \((B, \overline{H})\) is called a polarization of \( K \), denoted by \( \overline{B} \).

For \((\phi_0, \ldots, \phi_n) \in K^{n+1} \setminus \{0\} \), we define
\[
h^\overline{B}(\phi_0, \ldots, \phi_n) := 
\sum_{\Gamma} \max_i \{-\text{ord}_\Gamma(\phi_i)\} \hat{\deg} \left( \hat{c}_1 \left( \overline{H} \big|_{\Gamma} \right)^d \right) 
+ \int_{\overline{B}(\mathbb{C})} \log \left( \max_i \{|\phi_i|\} \right) c_1(\overline{H})^d.
\]
(\( \Gamma \)'s run over all prime divisors on \( B \))

It is easy to see
\[
h^\overline{B}(x\phi_0, \ldots, x\phi_n) = h^\overline{B}(\phi_0, \ldots, \phi_n).
\]
Thus we get
\[
h^\overline{B} : \mathbb{P}^n(K) \to \mathbb{R}.
\]

* In the case where \( K \) is a number field, \( h^\overline{B} \) is the arithmetic height function.

* In the case where \( B \) is an arithmetic surface and \( \overline{H} = (\mathcal{O}_B, c \cdot |\text{can}|) \) (\( 0 < c < 1 \)), \( h^\overline{B} \) is a constant multiple of the geometric height function as
6. Another Description

* Fix a polarization:

\[ K : \text{an arithmetic function field} \]
\[ d := \text{tr.deg}_{Q}(K). \]
\[ B : \text{a projective and flat integral scheme over } \mathbb{Z} \text{ whose function field is } K. \]
\[ \overline{H} : \text{a nef hermitian line bundle on } B. \]
\[ \overline{B} = (B, \overline{H}) : \text{a polarization of } K. \]

* Variety and line bundle over \( K \)

\[ X : \text{a projective variety over } K. \]
\[ L : \text{a line bundle on } X. \]

* Model of \((X, L)\)

\[ \mathcal{X} : \text{an integral projective scheme over } B \]
\[ \text{whose generic fiber of } \mathcal{X} \rightarrow B \text{ is } X. \]
\[ \overline{\mathcal{L}} : \text{a hermitian line bundle on } \mathcal{X} \text{ which gives rise to } L \text{ on the generic fiber of } \mathcal{X} \rightarrow B. \]

A pair \((\mathcal{X}, \overline{\mathcal{L}})\) is called a model of \((X, L)\).

* \( \Delta_{P} \) for \( P \in X(\overline{K}) \)

For \( P \in X(\overline{K}) \), the Zariski closure of the image

\[ \text{Spec}(\overline{K}) \overset{P}{\longrightarrow} X \hookrightarrow \mathcal{X} \]

is denoted by \( \Delta_{P} \).

Then we define \( h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}} : X(\overline{K}) \rightarrow \mathbb{R} \) to be

\[ h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(P) := \frac{\text{deg}\left(\hat{c}_{1}(\overline{\mathcal{L}}|_{\Delta_{P}}) \cdot \hat{c}_{1}(f^{*}(\overline{H})|_{\Delta_{P}})^{d}\right)}{[K(P) : K]}, \]

where \( f \) is the canonical morphism \( \mathcal{X} \rightarrow B \). Note that if \((\mathcal{X}', \overline{\mathcal{L}}')\) is another model of \((X, L)\), then there is a constant \( C \) with

\[ \left|h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(P) - h_{(\mathcal{X}', \overline{\mathcal{L}}')}^{\overline{B}}(P)\right| \leq C \quad (\forall P \in X(\overline{K})) \]
This means that $h_{(\mathcal{X},\overline{L})}^{\overline{B}}$ is uniquely determined modulo bounded functions on $X(\overline{K})$, so that we may write it as $h_{L}^{\overline{B}}$.

7. NORTHCOTT'S THEOREM

Theorem 1 (Northcott's theorem). We assume that $\overline{H}$ is big, i.e., $\text{rk}_{\mathbb{Z}}H^{0}(B, O(m^{d}))$ and for a sufficient large $n$, there is a non-zero $s \in H^{0}(B, H^{\otimes n})$ with $\|s\|_{\text{sup}} < 1$. Then, for any $M$ and $e$, the set

$$ \{ P \in X(\overline{K}) \mid h_{L}^{\overline{B}}(P) \leq M, \ [K(P) : K] \leq e \} $$

is finite.

Theorem 2 (Refinement). We assume that $\overline{H}$ is big. Then, for a fixed $e$,

$$ \log \# \{ P \in X(\overline{K}) \mid h_{L}^{\overline{B}}(P) \leq h, [K(P) : K] \leq e \} \frac{\log n}{h^{d+1}} $$

is bounded above as $h$ goes to the infinity.

8. THE NUMBER OF ALGEBRAIC CYCLES

In the similar techniques, we have the following:

Theorem 3 (Geometric version). Let $X$ be a projective scheme over a finite field $\mathbb{F}_{q}$ and $H$ a very ample line bundle on $X$. For a non-negative integer $k$, we denote by $n_{k}(X, H, l)$ the number of effective $l$-dimensional cycles with

$$ \deg(H^{l} \cdot V) = k. $$

Then, there is a constant $C$ depending only on $l$ and $\dim_{\mathbb{F}_{q}}H^{0}(X, H)$ such that

$$ \log_{q}(n_{k}(X, H, l)) \leq Ck^{l+1} $$

for all $k \geq 1$.

Theorem 4 (Arithmetic version). Let $X$ be a projective and flat integral scheme over $\mathbb{Z}$ and $\overline{H}$ an ample $C^{\infty}$-hermitian line bundle $X$. For a real number $h$, we denote by $n_{\leq h}(X, \overline{H}, l)$ the number of effective $l$-dimensional cycles with

$$ \overline{\deg}(\hat{c}_{1}(H)^{l} \cdot V) \leq h. $$

Then, there is a constant $C$ such that

$$ \log(n_{\leq h}(X, \overline{H}, l)) \leq Ch^{l+1} $$

for all $h \geq 1$. 
Remark 5. The above two theorems might give rise to new zeta functions. For example, in Theorem 3, if we set
\[ Z(X, H, l)(T) = \sum_{k=0}^{\infty} n_k(X, H, l)T^{k^{l+1}}, \]
then \( Z(X, H, l) \) is a convergent power series at 0. Moreover, in Theorem 4, if we set
\[ \zeta(X, \overline{H}, l)(s) = \sum_{V} \exp(-s \cdot \deg(\hat{c}_1(H) \cdot V)^{l+1}) \]
is a convergent Dirichlet series on \( \text{Re}(s) \gg 0 \), where \( V \) runs over all effective \( l \)-dimensional cycles.

9. Height Function on an Abelian Variety

We assume that \( X \) is an abelian variety \( A \). Let \( L \) be a symmetric ample line bundle on \( A \). Then, as in the usual theory of height functions, we have the canonical quadratic function
\[ \hat{h}_L^\overline{B} : A(\overline{K}) \to \mathbb{R}. \]
Actually, it is defined by
\[ \hat{h}_L^\overline{B}(P) := \lim_{n \to \infty} \frac{h_L^\overline{B}(nP)}{n^2}. \]
By Northcott’s theorem, if \( \overline{H} \) is big, then
\[ \hat{h}_L^\overline{B}(P) = 0 \iff P \in A(\overline{K})_{\text{tor}}. \]
From now on, we assume that \( \overline{H} \) is big. Here we set
\[ \langle x, y \rangle_L^\overline{B} = \frac{1}{2} \left( \hat{h}_L^\overline{B}(x + y) - \hat{h}_L^\overline{B}(x) - \hat{h}_L^\overline{B}(y) \right) \]
Then, \( \langle , \rangle_L^\overline{B} \) gives rise to an inner product \( A(\overline{K}) \otimes \mathbb{R} \). For \( x_1, \ldots, x_l \in A(\overline{K}) \), we set
\[ \delta_L^\overline{B}(x_1, \ldots, x_l) := \det \left( \langle x_i, x_j \rangle_L^\overline{B} \right). \]

10. Bogomolov + Mordell

Theorem 6. Let \( \Gamma \) be a subgroup of finite rank in \( A(\overline{K}) \), and \( Y \) a subvariety of \( A_{\overline{K}} \). Let us fix a basis \( \{\gamma_1, \ldots, \gamma_n\} \) of \( \Gamma \otimes \mathbb{Q} \). If the set
\[ \{x \in Y(\overline{K}) \mid \delta_L^\overline{B}(\gamma_1, \ldots, \gamma_n, x) \leq \epsilon \} \]
is Zariski dense in \( Y \) for every positive number \( \epsilon \), then \( Y \) is a translation of an abelian subvariety of \( A_{\overline{K}} \) by an element of \( \Gamma_{\text{div}} \), where
\[ \Gamma_{\text{div}} = \{x \in A(\overline{K}) \mid \exists n \in \mathbb{Z}_{>0} nx \in \Gamma \}. \]
Corollary 7 (Bogomolov's conjecture). Let $Y$ be a subvariety of $A_{\overline{K}}$. If the set
$$\{x \in Y(\overline{K}) \mid \hat{h}_{L}^{\overline{B}}(x) \leq \epsilon\}$$
is Zariski dense in $Y$ for every positive number $\epsilon$, then $Y$ is a translation of an abelian subvariety of $A_{\overline{K}}$ by a torsion point.

Corollary 8 (Mordell-Lang conjecture). Let $A$ be a complex abelian variety, $\Gamma$ a subgroup of finite rank in $A(\mathbb{C})$, and $Y$ a subvariety of $A$. Then, there are abelian subvarieties $C_1, \ldots, C_n$ of $A$, and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that
$$Y(\mathbb{C}) \cap \Gamma = \bigcup_{i=1}^{n} (C_i + \gamma_i)$$
and
$$Y(\mathbb{C}) \cap \Gamma = \bigcup_{i=1}^{n} (C_i(\mathbb{C}) + \gamma_i) \cap \Gamma.$$
Let $F$ be a finite extension of $K$. For $x \in A(\overline{K})$, we set

$O_F(x) := \{\sigma(x) \mid \sigma \in \mathrm{Gal}(\overline{K}/F)\}$.

For an integer $n \geq 2$, let $\beta_n : A^n \to A^{n-1}$ be a homomorphism given by

$\beta_n(x_1, \ldots, x_n) = (x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1)$.

For a subset $T$ of $S$ and a finite extension $F$ of $K$, we set

$D_n(T, F) = \bigcup_{s \in T} \beta_n(O_F(s)^n)$.

Moreover, we denote by $\overline{D}_n(T, F)$ the Zariski closure of $D_n(T, F)$.

A pair $(S, K)$ is said to be minimized if

1. for any infinite subset $T$ of $S$ and any finite extension $F$ of $K$,
   $\overline{D}_2(T, F) = \overline{D}_2(S, K)$;
2. $\overline{D}_2([N](S), K) = \overline{D}_2(S, K)$ for all integers $N \geq 1$.

Note that if an infinite subset $S$ of $A(\overline{K})$ is small with respect to $\Gamma$, then there are an infinite subset $T$ of $S$, a finite extension $F$ of $K$, and a positive integer $N$ such that $([N](T), F)$ is minimized.

**Theorem 9** (Poonen-Moriwaki). *Let $S$ be an infinite subset of $A(\overline{K})$ such that $S$ is small with respect to $\Gamma$. If $(S, K)$ is minimized, then there is an abelian subvariety $C$ of $A_{\overline{K}}$ such that $\overline{D}_n(S, K) = C^{n-1}$ for all $n \geq 2$.*

The above theorem is a consequence of Bogomolov’s conjecture.

Three ingredients:

1. the above theorem
2. the special case of Mordell-Lang conjecture
3. a geometric trick to remove a measure-theoretic argument in Poonen’s paper

imply the main theorem.

More precisely, we can prove it in the following way:

Replacing $K$ by a finite extension of $K$, we may assume that there is a finitely generated subgroup $\Gamma_0$ of $\Gamma \cap A(K)$ with $\Gamma_0 \otimes \mathbb{Q} = \Gamma \otimes \mathbb{Q}$. We set

$\mathrm{Stab}(Y) = \{a \in A \mid Y + a = Y\}$.

Considering $A/\mathrm{Stab}(Y)$, it is sufficient to show the following claim.

**Claim:** If $\mathrm{Stab}(Y) = \{0\}$, then $Y$ is a point.

We assume that $\dim Y > 0$. Then, replacing $K$ by a finite extension of $K$, we can find an infinite subset $S$ of $Y(\overline{K})$ with the following properties:

1. $S$ is small with respect to $\Gamma_{\mathrm{div}}$.
2. $S$ is Zariski dense in $Y$.
(3) $(S, K)$ is minimized.

Then, there is an abelian subvariety $C$ of $A_{\overline{K}}$ with $\overline{D}_n(S, K) = C^{n-1}$ for all $n \geq 2$. If $\dim C = 0$, then $S \subseteq A(K)$. Thus, by the special case of Mordell-Lang conjecture, $Y$ is a translation of an abelian subvariety $B$ of $A_{\overline{K}}$. Then, $\text{Stab}(Y) = B$. Thus, $\dim B = 0$, which implies $\dim Y = 0$, so that we have a contradiction.

Next we assume that $\dim C > 0$. Let us fix a positive integer $n$ with $n > 2 \dim(A)$. Let $\pi : A \to A/C$ be the natural homomorphism and $T = \pi(Y)$. Let $Y^n_T$ be the fiber product over $T$ in $Y^n$. Then, we have a morphism $\beta_n : Y^n_T \to A^{n-1}$ given by

$$\beta_n(x_1, \ldots, x_n) = (x_2 - x_1, \ldots, x_n - x_1).$$

Since $O_K(s)^n \subseteq X^n_T$, let $Y$ be the Zariski closure of $\bigcup_{s \in S} O_K(s)^n$. Then, $\beta_n(Y) \supseteq C^{n-1}$. Thus, we get

$$\dim(X^n_T) \geq \dim(C^{n-1}).$$

On the other hand, since $\text{Stab}(Y) = \{0\}$,

$$\dim(X/T) \leq \dim(C) - 1.$$

Thus,

$$\dim(X^n_T) - \dim(C^{n-1}) = (n \dim(X/T) + \dim(T)) - (n - 1) \dim(C)$$

$$\leq \dim(C) + \dim(T) - n$$

$$\leq 2 \dim(A) - n < 0.$$

This is a contradiction.

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