Bifurcation of the Kolmogorov flow with an external friction

東京理科大学理工学部 松田 真実 (Mami Matsuda)
Faculty of Science and Technology,
Tokyo University of Science

1 Introduction

Through my graduate school days studying under professor Sadao Miyatake, I have considered some bifurcation problems about the Kolmogorov flow. The Kolmogorov flow means a plane periodic flow of an incompressible fluid under the action of a spatially periodic external force. Since proposed in 1959, it has been conceived of only as a convenient object for theoretical investigations. But twenty years later, the flow was realized physically as a laboratory model by Bondarenko and his group (see its outline in [2] and Obkuhov[9]). The results of their experiments were found to be in good qualitative agreement with the previous theories described in Meshalkin and Sinai[8] and Iudovich[4], but in some cases, probably because they could only create a thin layer, there were some serious disagreement caused by a friction on the bottom of the channel. Then, they asserted that they should understand the influence of the friction in order to investigate a motion in a thin layer and built an updated model of the Kolmogorov flow with an external friction.

The corresponding equations in stationary case take the form:

\[
\begin{align*}
 uu_x + vu_y &= -P_x + \nu \Delta u - \kappa u + \gamma \sin y, \\
uu_x + vv_y &= -P_y + \nu \Delta v - \kappa v, \\
u_x + vv_y &= 0, \quad \text{in } \mathbb{R}^2,
\end{align*}
\]

where \( u = u(x, y) \) and \( v = v(x, y) \) are the velocity components, \( P = P(x, y) \) is the pressure, \( \nu > 0 \) is the kinematic viscosity, \( \gamma \) is the intensity of the external force (\( \gamma \sin y, 0 \)), \( \Delta \) is the two-dimensional Laplace operator, and \( \kappa \) is the coefficient of external friction.
which can be defined by the formula $\kappa \equiv 2\nu/h^2$ with $h$, the depth of the fluid layer. Let the system of solutions $V(x, y) = (u(x, y), v(x, y))$ and $P(x, y)$ satisfy

$$
\begin{align*}
V(x, y) &= V(x + 2\pi/\alpha, y) = V(x, y + 2\pi), \\
P(x, y) &= P(x + 2\pi/\alpha, y) = P(x, y + 2\pi), \\
\iint_D V(x, y) dxdy &= 0, \quad \iint_D P(x, y) dxdy = 0,
\end{align*}
$$

where $D = \{(x, y) : |x| \leq \pi/\alpha, |y| \leq \pi\}$.

Introducing the stream function $\psi(x, y)$, we represent the velocity as $(u, v) = (\psi_y, -\psi_x)$. The pressure is known to be determined by the velocity. Then, eliminating $P$ and replacing $\psi$ with $\gamma\nu^{-1}\psi$, we reduce the problem (1.1-2) to:

$$
(1.3) \quad \lambda J(\Delta\psi, \psi) = \nu\Delta^2\psi - \zeta\Delta\psi + \cos y,
$$

$$
\begin{align*}
\psi(x, y) &= \psi(x + 2\pi/\alpha, y) = \psi(x, y + 2\pi), \\
\iint_D \psi(x, y) dxdy &= 0,
\end{align*}
$$

where $\lambda \equiv \gamma/\nu^2$ and $\zeta \equiv \kappa/\nu = 2/h^2$.

We can see that $\psi_0(x, y) \equiv -(1 + \zeta)^{-1}\cos y$ satisfies (1.3-4) for any $\lambda > 0$ and $\zeta \geq 0$. We call this a basic solution. The velocity field of the basic solution is given by $(u_0, v_0) = (\gamma\nu^{-1}(1 + \zeta)^{-1}\sin y, 0)$, which represents a shear flow parallel to the $x$-axis.

We would like to search solutions in the form $\psi = \psi_0 + \varphi$. From (1.3), we have

$$
(1.5) \quad f(\lambda, \varphi) \equiv \left\{ \Delta^2 - \zeta\Delta - \lambda(1 + \zeta)^{-1}\sin y(\Delta + I)\partial_x \right\} \varphi - \lambda J(\Delta\varphi, \varphi) = 0,
$$

where $I$ is the identity operator. $\varphi = 0$ corresponds to the basic solution for all $\lambda$ and $\zeta$. We consider $\varphi$ in the Sobolev space $X$ satisfying (1.4) such as $X \equiv H^4(D)/R$ with the inner product defined by

$$(\varphi, \varphi)_X \equiv (\Delta^2\varphi, \Delta^2\varphi)_{L^2} < \infty, \quad \varphi \in X.$$
The problem is reduced the same one studied in [7] if \( \zeta = 0 \). As for this case where there's no external friction, professor Sadao Miyatake and myself have examined the bifurcation curves of solutions to the problem with a symmetric condition \( \varphi(x, y) = \varphi(-x, -y) \) in order to use Crandall-Rabinowitz bifurcation theorem which requires 
\[ \dim \ker f_{\varphi}(\lambda_0, 0) = 1. \]
However, in this time we first remove the symmetric condition for the velocity, then obtain the similar result as seen in [7].

2 Guideline of the proof

2.1 Linearized equations

First, we solve the linearized equation and obtain the function \( \lambda = \lambda(\beta, \zeta) \) defined on \( \beta \in (0, 1) \) and \( \zeta \in [0, \infty) \). The linearized eigenvalue problem for fixed \( \alpha \) and \( \zeta \) is

\[ f_{\varphi}(\lambda, 0)\varphi = \{ \Delta^2 - \zeta \Delta - \lambda(1 + \zeta)^{-1} \sin y(\Delta + I)\partial_x \} \varphi = 0, \]

where \( \lambda \) is called eigenvalue if (2.1) has a solution \( \varphi \neq 0 \).

\( \varphi \in X \) is expanded in the Fourier series:

\[ \varphi = \sum_{m,n} c_{m,n} e^{i(m\alpha x + ny)}, \quad \sum_{m,n} (m^2 \alpha^2 + n^2)^4 |c_{m,n}|^2 < +\infty, \quad c_{0,0} = 0, \]

where the summation is taken over all the pairs of integers but \( (m, n) = (0, 0) \). \( c_{0,0} = 0 \) follows from \( \int_D \varphi dx dy = 0 \).

For each integer \( m \), the coefficients \( c_{m,n} \) satisfy the infinite system of linear equations:

\[ (m^2 \alpha^2 + n^2 - 1)c_{m,n} - \frac{\lambda m\alpha}{2(1+\zeta)} \{ m^2 \alpha^2 + (n+1)^2 - 1 \} c_{m,n+1} = 0, \quad n = 0, \pm 1, \pm 2, \cdots \]

We see \( c_{0,n} = 0 \) for any integer \( n \). For \( m \neq 0 \), we put

\[ a_{m,n} \equiv \frac{2(1+\zeta)(m^2 \alpha^2 + n^2)(m^2 \alpha^2 + n^2 + \zeta)}{\lambda m\alpha(m^2 \alpha^2 + n^2 - 1)}, \quad b_{m,n} \equiv (m^2 \alpha^2 + n^2 - 1)c_{m,n}, \]

then the above equations are simply described by

\[ a_{m,n} b_{m,n} + b_{m,n-1} - b_{m,n+1} = 0, \quad n = 0, \pm 1, \pm 2, \cdots \]
We remark that the set of solutions \( \{ b_{m,n} \} \) is one dimensional. Let us seek non-trivial solutions of the system (2.2) such that \( b_{m,n} \to 0 \) as \( |n| \to \infty \) for each \( m \neq 0 \). In order to find these \( b_{m,n} \), we need to solve the following equation:

\[
-\frac{a_{m,0}}{2} = \frac{1}{a_{m,1}} + \frac{1}{a_{m,2}} + \cdots.
\]

We may restrict ourselves to the case where \( m > 0 \), since for negative \( m \) the argument is similar because of \( a_{m,n} = -a_{-m,n} \). We omit \( m \) and put \( \beta \equiv m \alpha \) and \( a_{n} \equiv a_{m,n} \) simply. Denoting the right hand side of (2.3) by \( G(\lambda, \beta, \zeta) \), we rewrite (2.3) as

\[
\frac{(1+\zeta)\beta(\beta^2+\zeta)}{\lambda(1-\beta^2)} = G(\lambda, \beta, \zeta).
\]

We state properties of (2.3') in the following proposition (the proof is written in [12]).

**Proposition 1** For the solutions of (2.3'), we obtain the following results:

1. (2.3') has no positive solution if \( \beta > 1 \) and \( \zeta \geq 0 \).
2. If \( 0 < \beta < 1 \), there exists a continuous function \( \lambda(\beta, \zeta) \) such that:
   
   i. (2.3') has a solution if and only if \( \lambda = \lambda(\beta, \zeta) \);
   
   ii. For fixed \( \zeta > 0 \), \( \lim_{\beta \to 0} \lambda(\beta, \zeta) = \lim_{\beta \to 1} \lambda(\beta, \zeta) = +\infty \) and for \( \zeta = 0 \), it holds \( \lim_{\beta \to 0} \lambda(\beta, 0) = \sqrt{2} \) and \( \lim_{\beta \to 1} \lambda(\beta, 0) = +\infty \);
   
   iii. For fixed \( \beta \in (0,1) \), \( \lambda(\beta, \zeta) \) is a strictly monotone increasing function of \( \zeta > 0 \).

Because of this difference between \( \zeta > 0 \) and \( \zeta = 0 \), Bondarenko and his groups created an updated model with an external friction.

From (2) of Proposition 1, (2.3) has a solution \( \lambda = \lambda(\beta, \zeta) \equiv \lambda_{k} \) only if \( \beta \equiv k\alpha \in (0,1) \). Then, integer \( k \) is restricted as follows:

\[
k \in K_{\alpha} \equiv \{1, 2, \ldots, r ; r \in N, r\alpha < 1 \leq (r+1)\alpha\}.
\]

Then, we take a solution \( b_{k,n} \) for \( k \in K_{\alpha} \) defined by

\[
b_{k,n} \equiv \begin{cases} 
\Pi_{i=1}^{n} \rho_{k,i} & \text{for } n > 0, \\
1 & \text{for } n = 0, \\
(-1)^{n} \Pi_{i=1}^{-n} \rho_{k,i} & \text{for } n < 0,
\end{cases}
\]

\[(2.4)\]
\[ \rho_{k,i} = \frac{-1}{a_{k,i}} + \frac{1}{a_{k,i+1}} + \cdots, \quad a_{k,i} = a_{k,i}(\lambda_k), \quad i \geq 1. \]

Let us consider the case where \( m < 0 \) and \( |m| \in K_\alpha \). As we note \( a_{m,n} = -a_{-m,n} \), we obtain that \( b_{-k,n} = (-1)^n b_{k,n} \) for \( k \in K_\alpha \) also satisfy (2.2). Therefore, the set of the non-trivial solutions of (2.1) is given as follows:

(2.5) \[ \ker f_\varphi(\lambda_k, 0) = \{ \varphi^{(k)} = t_1 \varphi_k + t_2 \varphi_{-k} ; \ t_1, t_2 \in \mathbb{R} \}, \]

where \( \varphi_k \equiv \sum_{n=-\infty}^{+\infty} c_{k,n} e^{(k\alpha x + ny)} \), \( c_{k,n} = (k^2 \alpha^2 + n^2 - 1)^{-1} b_{k,n} \). We see that \( \varphi_{-k} \) is equal to \( \overline{\varphi}_k \), the conjugate function of \( \varphi_k \), since we have \( c_{-k,n} = (-1)^n c_{k,n} = c_{k,-n} \) due to \( b_{-k,n} = (-1)^n b_{k,n} = b_{k,-n} \). Moreover, using Euler’s formula, we can rewrite (2.5):

(2.5') \[ \ker f_{\varphi}(\lambda_k, 0) = \{ \varphi^{(k)} = s_1 \varphi_{k,1} + s_2 \varphi_{k,2} ; \ s_1, s_2 \in \mathbb{R} \}, \]

where \( \varphi_{k,1} \equiv \sum_{n=-\infty}^{\infty} d_{k,n} \cos(k\alpha x + ny) \) and \( \varphi_{k,2} \equiv \sum_{n=-\infty}^{\infty} d_{k,n} \sin(k\alpha x + ny) \).

Similarly, let us seek non-trivial solutions \( \Phi \) of the conjugate equation of (2.1):

(2.6) \[ f_{\varphi}^*(\lambda, 0) \Phi = \{ \Delta^2 - \zeta \Delta + \lambda (1 + \zeta)^{-1} (\Delta + I) \sin y \partial_x \} \Phi = 0, \]

in the form \( \Phi(x, y) = \sum_{m,n} d_{m,n} e^{i(m\alpha x + ny)} \). \( f_{\varphi} \) is a bounded operator from \( H_0^t \) to \( H_0^{t-4} \). And we have the following relation of \( d_{m,n} \) for each integer \( m \):

\[ a_{m,n} d_{m,n} - d_{m,n-1} + d_{m,n+1} = 0. \]

Putting \( b_{m,n}' \equiv (-1)^n d_{m,n} \), we have also

\[ a_{m,n} b_{m,n}' + b_{m,n-1}' - b_{m,n+1}' = 0, \]

which is the same form as (2.2). Applying the same argument as that in (2.2), we obtain the non-trivial solutions of (2.6) if \( \lambda = \lambda_k k \in K \):

(2.7) \[ \ker f_{\varphi}(\lambda_k, 0) = \{ \Phi^{(k)} = t_1 \Phi_k + t_2 \Phi_{-k} ; \ t_1, t_2 \in \mathbb{R} \}, \]

where \( \Phi_k = \sum_{n=-\infty}^{+\infty} d_{k,n} e^{i(k\alpha x + ny)} \), \( d_{k,n} = (-1)^n b_{k,n} \) and \( b_{k,n} \) are given by (2.4). Note that each \( \Phi^{(k)} \) in \( \ker f_{\varphi}(\lambda_k, 0) \) is smooth function. We rewrite \( \Phi^{(k)} \) in \( \ker f_{\varphi}(\lambda_k, 0) \) as

(2.7') \[ \ker f_{\varphi}(\lambda_k, 0)^* = \{ \Phi^{(k)} = s_1 \Phi_{k,1} + s_2 \Phi_{k,2} ; \ s_1, s_2 \in \mathbb{R} \}, \]

where \( \Phi_{k,1} \equiv \sum_{n=-\infty}^{+\infty} d_{k,n} \cos(k\alpha x + ny) \) and \( \Phi_{k,2} \equiv \sum_{n=-\infty}^{+\infty} d_{k,n} \sin(k\alpha x + ny) \).

We remark that the both \( \ker f_{\varphi}(\lambda_k, 0) \) and \( \ker f_{\varphi}^*(\lambda_k, 0) \) are two dimensional spaces.
2.2 Existence of bifurcation points

For \( \alpha \in (0, 1) \) and \( \zeta \in [0, \infty) \), (2.1) has non-trivial solutions if and only if \( \lambda \) is equal to the values \( \lambda_k \) given in the previous section. Using the method of Ljapunov-Schmidt, we prove that \( \lambda = \lambda_k \) is the bifurcation point of (1.5).

Assume \( \varphi \in X \) and \( \omega \in Y \equiv L_0^2 \) where \( g \in L_0^2 \) means \( g \in L^2 \) and \( \iint_D g \, dx \, dy = 0 \). We decompose them orthogonaly by:

\[
\begin{align*}
\varphi &= \varphi_1 + \varphi_2, \quad \varphi_1 \in X_1, \quad \varphi_2 \in X_2, \\
\omega &= \omega_1 + \omega_2, \quad \omega_1 \in Y_1, \quad \omega_2 \in Y_2.
\end{align*}
\]

\( X_i \) and \( Y_i \) \((i = 1, 2)\) are defined as follows: \( X_1 = \ker f_{\varphi}(\lambda_k, 0) \), \( X_2 \) is the orthogonal complement of \( X_1 \), \( Y_2 \) is the range of \( f_{\varphi}(\lambda_k, 0) \) and \( Y_1 \) is the orthogonal complement of \( Y_2 \).

According to Section 2, \( X_1 = \ker f_{\varphi}(\lambda_k, 0) \) and \( \ker f_{\varphi}^{*}(\lambda_k, 0) \) are two dimensional space. We also see \( \dim Y_1 \) is two, namely, we verify

\[ Y_1 = \ker f_{\varphi}^{*}(\lambda_k, 0). \]

In fact, put \( T \equiv f_{\varphi}(\lambda_k, 0) \) and \( T^* \equiv f_{\varphi}^*(\lambda_k, 0) \), then \( \omega_1 \in Y_1 \) satisfies \((\omega_1, T\psi)_{L^2} = 0 \) for \( \psi \in X \). Hence we have \( T^*\omega_1 = 0 \) in the sense of distribution. Although \( \omega_1 \) belongs to \( L_0^2 \) space and \( \ker T^* \) is subspace of \( X = H_0^4 \), we can see that this \( \omega_1 \) is smooth enough to belong to \( \ker T^* \) by the hypo-ellipticity as follows. From (2.6), we write \( T^* = \Delta^2 + T^{(3)} \). Then \( T^*\omega_1 = 0 \) implies \( \Delta^2 \omega_1 = -T^{(3)}\omega_1 \). Since \( \omega_1 \in Y_1 \), the right hand-side of this equation belongs to \( H_0^{(-3)} \), namely, the Fourier expansion coefficients of \( \omega_1 \) satisfy \( \sum(m^2 + n^2)^{-3}c_{m,n}^2 < \infty \). Then the left hand-side belongs to \( H_0^{(-3)} \), which implies \( \omega_1 \in H_1 \). Repeating this several times, we see that \( \omega_1 \) is sufficiently smooth.

We denote the projection to \( Y_1 \) of \( Y \) by \( P \). Then, \( Q \equiv I - P \) is the projection to \( Y_2 \). Corresponding to the above decomposition, we have the system of the following two equations which is equivalent to (1.5):

\[
\begin{align*}
Qf(\lambda, \varphi_1 + \varphi_2) &= 0 \quad \text{in } Y_2, & \cdots \quad (3.2) \\
Pf(\lambda, \varphi_1 + \varphi_2) &= 0 \quad \text{in } Y_1. & \cdots \quad (3.3)
\end{align*}
\]

Hereafter, we seek the solution \((\lambda, \varphi)\) of this system, depending on one parameter \( s \in (-1, 1) \) as follows: \((\lambda, \varphi) = (\mu(s), \varphi_1(s) + \varphi_2(s))\). We suppose that \( \mu(s) \in R \), \( \varphi_1(s) \in X_1 \) and \( \varphi_2(s) \in X_2 \) satisfy \( \mu(0) = \lambda_k \). We put \( \varphi_1(s) = s\varphi^{(k)} \) where \( \varphi^{(k)} \) is a non-trivial solution of (2.1) given in (2.5). Then we look for \( \lambda = \mu(s) \) and \( \varphi_2(s) \).
First, let us consider (3.2). We put $Qf(\lambda, \varphi_1 + \varphi_2) \equiv g(\tau, \varphi_2)$ with $\tau \equiv (\lambda, s)$ for fixed $\alpha \in (0, 1)$ and $\zeta \in [0, \infty)$. Note that $g(\tau_k, 0) = 0$ for $\tau_k \equiv (\lambda_k, 0)$ since $f(\lambda, 0) = 0$. By definition we see that $g_{\varphi_2}(\tau_k, 0) = Qf_{\varphi}(\lambda_k, 0)$ is a bijective mapping from $X_2$ to $Y_2$.

Then from the implicit function theorem, there exists a function $\psi(\tau)$ which satisfies $g(\tau, \psi(\tau)) = 0$ and $\psi(\tau_k) = 0$ in the neighborhood of $(\tau_k, 0)$. We shall determine $\psi = \psi(\tau)$ more precisely. From (3.2), with $\varphi_1 = s\varphi^{(k)}$ and $\varphi_2 = \psi$, $\psi$ satisfies the following equation:

$$H[\psi] - \tilde{L}[s\varphi^{(k)} + \psi] - \lambda J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi) = 0,$$

where $H \equiv Qf_{\varphi}(\lambda_k, 0)$, $\tilde{L} \equiv (\lambda - \lambda_k)(1 + \zeta)^{-1}\sin y(\Delta + I)\partial_x$. Since $H$ is a bijective mapping from $X_2$ to $Y_2$, it holds that

$$\psi - H^{-1}\tilde{L}[s\varphi^{(k)} + \psi] - \lambda H^{-1}J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi) = 0.$$

We define a sequence of functions $\{\psi_n\} (n = 0, 1, 2, \cdots)$ as follows:

$$\psi_0 = 0, \quad \psi_n \equiv H^{-1}\tilde{L}[s\varphi^{(k)} + \psi_{n-1}] - \lambda H^{-1}J(\Delta(s\varphi^{(k)} + \psi_{n-1}), s\varphi^{(k)} + \psi_{n-1}).$$

Let us show that $\{\psi_n\}$ is a Cauchy sequence in the neighborhood of $s = 0$. In fact, since the non-linear term becomes $O(s^2)$, it can be omitted. Choosing $\lambda$ such as $|\lambda - \lambda_k| \leq 4^{-1}\|H^{-1}\|^{-1}$, we have $\|\psi_1\| = O(s)$ and $\|\psi_2 - \psi_1\| \leq 2^{-1}\|\psi_1\|$. Similarly, it holds that $\|\psi_{n+1} - \psi_n\| \leq 2^{-n}\|\psi_1\|$. Then $\{\psi_n\}$ is a Cauchy sequence and converges to a limit $\psi = \psi(\lambda, s)$ which belongs to $X_2$ satisfying $\psi(\lambda, 0) = 0$ and

$$\psi = H^{-1}\tilde{L}[s\varphi^{(k)} + \psi] - \lambda H^{-1}J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi)$$

for small $s$.

In order to show that $\lambda_k$ is a bifurcation point, we have to prove the existence of the solution $\mu(s)$ of (3.3) satisfying $\mu(0) = \lambda_k$. Substituting $\varphi_2 = \psi(\tau)$ into the left hand side of (3.3) and defining

$$Pf(\lambda, s\varphi^{(k)} + \psi(\lambda, s)) \equiv h(\lambda, s),$$

we denote

$$\chi(\lambda, s) \equiv \left\{ \begin{array}{ll}
\{h(\lambda, s) - h(\lambda, 0)\}/s, & \text{for } s \neq 0, \\
h_\ast(\lambda, 0), & \text{for } s = 0.
\end{array} \right.$$  

Note that $h(\lambda, 0) = 0$ holds and the continuity of $\chi$ follows from that of $h_\ast$. The reason why we define $\chi(\lambda, s)$ is that we cannot apply the implicit function theorem to
Remark that $h_{\lambda}(\lambda, 0) = 0$ holds from $\psi(\lambda, 0) = 0$ for all $\lambda$. From $h_{s}(\lambda, s) = P_{f_{\varphi}}(\lambda, s\varphi^{(k)} + \psi(\lambda, s)\right)\varphi^{(k)} + \psi_{s}(\lambda, s)$, it holds that $h_{s}(\lambda, 0) = P_{f_{\varphi}}(\lambda, 0)[\varphi^{(k)} + \psi_{s}(\lambda, 0)]$.

Now we verify $\psi_{s}(\lambda, 0) = 0$. Differentiating $Qf(\lambda, s\varphi^{(k)} + \psi(\lambda, s)) = 0$ by $s$ and putting $(\lambda, s) = (\lambda_{k}, 0)$, we have $Qf_{\varphi}(\lambda_{k}, 0)[\psi_{s}(\lambda_{k}, 0)] = 0$. Since $Qf_{\varphi}(\lambda_{k}, 0)$ is a bijective mapping from $X_{2}$ to $Y_{2}$, $\psi_{s}(\lambda_{k}, 0) = 0$ holds.

$\chi(\lambda, s) = 0$ is equivalent to the following equations:

(3.5) \[ \chi^{(1)}(\lambda, s) \equiv (\chi(\lambda, s), \Phi_{k,1})_{L^{2}} = 0, \]

(3.6) \[ \chi^{(2)}(\lambda, s) \equiv (\chi(\lambda, s), \Phi_{k,2})_{L^{2}} = 0, \]

where $\Phi_{k,i} \in Y_{1} = \ker f_{\varphi}^{*}(\lambda_{k}, 0)$ ($i = 1, 2$). First, we seek a solution $\lambda$ of (3.5) putting $\varphi^{(k)} = t_{1}\varphi_{k,1} + t_{2}\varphi_{k,2}$ for $(t_{1}, t_{2}) \neq (0, 0)$. Differentiating (3.5) by $\lambda$, then we have

\[ \chi^{(1)}_{\lambda}(\lambda_{k}, 0) = \left( \lim_{\Delta \lambda \rightarrow 0} \frac{\chi(\lambda_{k} + \Delta \lambda, 0) - \chi(\lambda_{k}, 0)}{\Delta \lambda}, \Phi_{k,1} \right)_{L^{2}} = (P_{f_{\varphi}}(\lambda_{k}, 0)[\varphi^{(k)}], \Phi_{k,1})_{L^{2}} = (f_{\varphi}(\lambda_{k}, 0)[\varphi^{(k)}], P^{*}\Phi_{k,1})_{L^{2}} = t_{1}(-1 + \zeta)^{-1}\sin y(\Delta + I)\partial_{x}\varphi_{k,1}, \Phi_{k,1})_{L^{2}}. \]

We show

(3.7) \[ (-1 + \zeta)^{-1}\sin y(\Delta + I)\partial_{x}\varphi_{k,1}, \Phi_{k,1})_{L^{2}} > 0. \]

Since $\varphi_{k,1}$ is a solution of (2.1), we have

\[ -(1 + \zeta)^{-1}\sin y(\Delta + I)\partial_{x}\varphi_{k,1} = \lambda_{k}^{-1}(\zeta)(-\Delta^{2} + \zeta\Delta)\varphi_{k,1}. \]

Using $\varphi_{k,1} = \sum_{n}c_{k,n}\cos(k\alpha x + ny)$ and $\Phi_{k,1} = \sum_{n}d_{k,n}\cos(k\alpha x + ny) = \sum_{n}(-1)^{n}(k^{2}\alpha^{2} + n^{2} - 1)c_{k,n}\cos(k\alpha x + ny)$, we obtain

\[ ((-\Delta^{2} + \zeta\Delta)\varphi_{k,1}, \Phi_{k,1})_{L^{2}} = \frac{1}{2}|D|\sum_{n}(-1)^{n+1}\tilde{c}_{k,n}, \]

where $\tilde{c}_{k,n} \equiv (k^{2}\alpha^{2} + n^{2})(k^{2}\alpha^{2} + n^{2} + \zeta)(k^{2}\alpha^{2} + n^{2} - 1)c_{k,n}^{2}$. Meanwhile, we can verify $\sum_{n}\tilde{c}_{k,n} = 0$ (seen in Iudovich[4]). In fact, from $f_{\varphi}(\lambda_{k}, 0)\varphi_{k,1} = 0$, multiplying this equation $(\Delta + I)\varphi_{k,1}$ and integrating over the rectangle $D$, we obtain

\[ 0 = \iint_{D}(\Delta + I)\varphi_{k,1}(\Delta^{2} - \zeta\Delta)\varphi_{k,1}dxdy - \lambda_{k}(1 + \zeta)^{-1}\iint_{D}(\Delta + I)\varphi_{k,1}\sin y(\Delta + I)\partial_{x}\varphi_{k,1}dxdy, \]
and see that the second term vanishes. Then, we have
\[
\int_D (\Delta + I) \varphi_{k,1} (\Delta^2 - \zeta \Delta) \varphi_{k,1} \, dx \, dy = \frac{-1}{2} |D| \sum_n \tilde{c}_{k,n} = 0.
\]
From \( \sum_n \tilde{c}_{k,n} = 0 \) and \( \tilde{c}_{k,-n} = \tilde{c}_{k,n} \), we obtain (3.7) since it holds
\[
\sum_n (-1)^{n+1} \tilde{c}_{k,n} = -\tilde{c}_{k,0} + 2 \sum_{m=1,3,5,\ldots} \tilde{c}_{k,m} - 2 \sum_{m=2,4,6,\ldots} \tilde{c}_{k,m} = 4 \sum_{m=1,3,5,\ldots} \tilde{c}_{k,m} > 0.
\]
As a result, we have \( \chi^{(1)}_\lambda (\lambda_k, 0) \neq 0 \) if \( t_1 \neq 0 \). From the implicit function theorem, there exists a function \( \lambda = \mu(s) \) satisfying \( \chi^{(1)}_\lambda (\mu(s), s) = 0 \) and \( \mu(0) = \lambda_k \).

Next, we suppose the question whether \( \lambda = \mu(s) \) satisfies (3.6). Since \( h_s(\lambda_k, 0) = 0 \) holds from \( h_s(\lambda, 0) = Pf_\varphi(\lambda, 0)[\varphi^{(k)} + \psi_s(\lambda, 0)] \) and \( \psi_s(\lambda_k, 0) = 0 \), we can see \( \chi^{(2)}(\lambda, 0) = (h_s(\lambda_k, 0), \Phi_{k,2})_{L^2} = 0 \). As for \( s \neq 0 \), it holds
\[
s \chi^{(2)}(\lambda, s) = (h(\lambda, s), \Phi_{k,2})_{L^2} = (Pf(\lambda, s\varphi^{(k)} + \psi(\lambda, s)), \Phi_{k,2})_{L^2} = (f(\lambda, s\varphi^{(k)} + \psi(\lambda, s)), \Phi_{k,2})_{L^2}.
\]
Then we have the following formula:
\[
s \chi^{(2)}(\mu(s), s) = (f(\mu(s), s\varphi^{(k)} + \psi(\mu(s), s)), \Phi_{k,2})_{L^2} = \left( \{\Delta^2 - \zeta \Delta - \mu(s) \sin y(\Delta + I) \partial_x \} [s\varphi^{(k)} + \psi(\mu(s), s)], \Phi_{k,2} \right)_{L^2} - \mu(s) \left( J(\Delta s\varphi^{(k)} + \psi(\mu(s), s)), s\varphi^{(k)} + \psi(\mu(s), s)), \Phi_{k,2} \right)_{L^2}.
\]
The question is how we choose \( \varphi^{(k)} \). From (3.4), if \( \varphi^{(k)} \) is represented as a linear combination of \( \varphi_{k,1} \) and \( \varphi_{k,2} \), \( \psi(\mu(s), s) \) is expanded by both sine and cosine functions. In this case, we cannot expect in general that the above formula goes to zero. However, if we put \( \varphi^{(k)} = \varphi_{k,1} \), \( \psi(\mu(s), s) \) is expanded by cosine only. As a result, the inner-product with \( \Phi_{k,2} \) becomes zero and, hence, \( \mu(s) \) satisfies (3.6). Thus, we obtain the former part of Theorem 1.
2.3 Properties of the Bifurcation curve

We shall consider the convex property of $\lambda = \mu(s)$ with regard to $s$. Putting $T = f_\varphi(\lambda, 0)$ and $\tilde{\lambda}(s) \equiv \mu(s) - \lambda_k$, we rewrite $f(\mu(s), \varphi(s)) = 0$ as

\[(4.1) \quad T \varphi(s) = \frac{\tilde{\lambda}(s)}{1 + \zeta} \sin y(\Delta + I) \partial_x \varphi(s) + \mu(s)J(\Delta \varphi(s), \varphi(s)), \]

where $\varphi(s) \equiv s \phi_{k,1} + \psi(\mu(s), s)$. Let us differentiate (4.1) by $s$:

\[
T \varphi_s(s) = \frac{\tilde{\lambda}_s(s)}{1 + \zeta} \sin y(\Delta + I) \partial_x \varphi(s) + \mu_s(s)J(\Delta \varphi(s), \varphi(s))_s + \frac{\tilde{\lambda}(s)}{1 + \zeta} \sin y(\Delta + I) \partial_x \varphi_s(s); \\
T \varphi_{ss}(s) = \frac{\tilde{\lambda}_{ss}(s)}{1 + \zeta} \sin y(\Delta + I) \partial_x \varphi(s) + \frac{2\tilde{\lambda}_s(s)}{1 + \zeta} \sin y(\Delta + I) \partial_x \varphi_s(s) \\
+ \frac{\tilde{\lambda}(s)}{1 + \zeta} \sin y(\Delta + I) \partial_x \varphi_{ss}(s) + \mu_{ss}(s)J(\Delta \varphi(s), \varphi(s))_s + \mu_s(s)J(\Delta \varphi(s), \varphi(s))_s; \\
\varphi_s(s) = \phi_{k,1} + \psi_\lambda(\mu(s), s) \mu_s(s) + \psi_s(\mu(s), s).
\]

Putting $s = 0$, we have

\[(4.2) \quad T \varphi_{ss}(0) = \frac{2\mu_s(0)}{1 + \zeta} \sin y(\Delta + I) \partial_x \varphi_{k,1} + 2\lambda_kJ(\Delta \varphi_{k,1}, \varphi_{k,1}). \]

If we take the $L^2$ inner-product with $\Phi_{k,1} \in \ker T^*$, (4.2) becomes

\[0 = \frac{2\mu_s(0)}{1 + \zeta} \langle \sin y(\Delta + I) \partial_x \varphi_{k,1}, \Phi_{k,1} \rangle_{L^2} + 2\lambda_k \langle J(\Delta \varphi_{k,1}, \varphi_{k,1}), \Phi_{k,1} \rangle_{L^2}, \]

and from $T \varphi_{k,1} = 0$, we obtain

\[0 = \frac{2\mu_s(0)}{\lambda_k} \langle (\Delta^2 - \zeta \Delta) \varphi_{k,1}, \Phi_{k,1} \rangle_{L^2} + 2\lambda_k \langle J(\Delta \varphi_{k,1}, \varphi_{k,1}), \Phi_{k,1} \rangle_{L^2}. \]

Since the Fourier coefficients of $J(\Delta \varphi_{k,1}, \varphi_{k,1})$ consist of a linear combination of $\cos ny$ and $\cos(2k\alpha x + ny)$, we have $\langle J(\Delta \varphi_{k,1}, \varphi_{k,1}), \Phi_{k,1} \rangle_{L^2} = 0$. Also, from the proof of (3.7), we have

\[(4.3) \quad \langle (\Delta^2 - \zeta \Delta) \varphi_{k,1}, \Phi_{k,1} \rangle_{L^2} < 0. \]
Therefore, we obtain $\mu_s(0) = 0$.

Differentiating (4.1) once more and putting $s = 0$, we have
\[
T \varphi_{sss}(0) = 3\mu_{ss}(0)(1 + \zeta)^{-1}\sin y(\Delta + I) \partial_x \varphi_{k,1} \\
+ 3\lambda_k \left\{ J(\Delta \varphi_{ss}(0), \varphi_{k,1}) + J(\Delta \varphi_{k,1}, \varphi_{ss}(0)) \right\} \\
= 3\mu_{ss}(0)\lambda_k^{-1}(\Delta^2 - \zeta\Delta)\varphi_{k,1} \\
+ 3\lambda_k \left\{ J(\Delta \varphi_{ss}(0), \varphi_{k,1}) + J(\Delta \varphi_{k,1}, \varphi_{ss}(0)) \right\},
\]
and taking the $L^2$ inner-product with $\Phi_{k,1} \in \ker T^*$,
\[
0 = (T \varphi_{sss}(0), \Phi_{k,1})_{L^2} \\
= 3\mu_{ss}(0)\lambda_k^{-1}((\Delta^2 - \zeta\Delta)\varphi_{k,1}, \Phi_{k,1})_{L^2} \\
+ 3\lambda_k (J(\Delta \varphi_{ss}(0), \varphi_{k,1}) + J(\Delta \varphi_{k,1}, \varphi_{ss}(0)), \Phi_{k,1})_{L^2}
\]
holds. Then we have
\[
\mu_{ss}(0) = \frac{-\lambda_k^2}{((\Delta^2 - \zeta\Delta)\varphi_{k,1}, \Phi_{k,1})_{L^2}} \left( J(\Delta \varphi_{ss}, \varphi_{k,1}) + J(\Delta \varphi_{k,1}, \varphi_{ss}), \Phi_{k,1} \right)_{L^2}.
\]
Let us determine the sign of $\mu_{ss}(0)$. From (4.3), this sign is equal to that of
\[
\int_D \left\{ J(\Delta \varphi_{ss}, \varphi_{k,1}) + J(\Delta \varphi_{k,1}, \varphi_{ss}) \right\} \Phi_{k,1} dxdy.
\]
Here $\varphi_{ss} \equiv \varphi_{ss}(0) = \psi_{ss}(\lambda_k, 0)$ is obtained by
\[
T \varphi_{ss} = 2\lambda_k J(\Delta \varphi_{k,1}, \varphi_{k,1}).
\]
The right-hand side of (4.5) consists of two terms extended respectively by $\cos \ell y$ and $\cos(2k\alpha x + \ell y)$.

We have the following proposition:

**Proposition 2** The solution of (4.5) takes the following form:
\[
\varphi_{ss} = \iota w^{(0)} A c(0) + \iota w^{(2k)} D E c(2k\alpha) \equiv Z_1 + Z_2, \\
Z_1 \equiv \iota w^{(0)} A c(0), \quad Z_2 \equiv \iota w^{(2k)} D E c(2k\alpha).
\]
Here $c(0)$, $c(2\alpha)$, $w^{(0)}$ and $w^{(2k)}$ are column vectors with the following $\ell$-th components:

$$(c(0))_\ell = \cos \ell y, \quad (c(2\alpha))_\ell = \cos(2\alpha x + \ell y),$$

$$(w^{(0)})_\ell = \lambda_k \alpha \ell \varphi^{(k)} K S^\ell \varphi^{(k)},$$

$$(w^{(2k)})_\ell = \lambda_k \alpha \ell \varphi^{(k)} K (2N - \ell I) RS^\ell \varphi^{(k)},$$

where $\varphi^{(k)}$ is a column vector corresponding to the Fourier coefficients of $\varphi_{k,1}$ with $n$-th component $\varphi_n = (k^2 \alpha^2 + n^2 - 1)^{-1} b_{k,n}$ ($b_{k,n}$ is defined by (2.6)), $K$ and $N$ are diagonal matrices with $n$-th elements $-\kappa_n$, and $S^\ell$ and $R$ are matrices with $(i, j)$ elements as follows:

$$(S^\ell)_{i,j} = \begin{cases} 
1 & \text{for } j-i = \ell, \\
0 & \text{otherwise},
\end{cases} \quad (R)_{i,j} = \begin{cases} 
1 & \text{for } i+j = 0, \\
0 & \text{otherwise}.
\end{cases}$$

$\Lambda$ and $E$ are diagonal matrices with $n$-th elements

$$\Lambda_n = \begin{cases} 
(n^4 + \zeta n^2)^{-1} & \text{for } n \neq 0, \\
0 & \text{for } n = 0,
\end{cases} \quad E_n = \frac{1+\zeta}{\lambda_k \alpha (4k^2 \alpha^2 + n^2 - 1)},$$

and $D = (\cdots d^{(m)} \cdots)$ is a matrix where $d^{(m)}$ are column vectors with $n$-th component $d^{(m)}_n$ as follows:

$$\begin{aligned}
d^{(m)}_n &= \begin{cases} 
\prod_{i=n+1}^{m+1} \eta_{i}^+ N_{m+1}^{-1} & \text{for } n > m, \\
N_{m+1}^{-1} & \text{for } n = m, \\
\prod_{i=n+1}^{m} \eta_{i}^- \eta_{m+1}^+ & \text{for } n < m,
\end{cases}
\end{aligned}$$

where

$$\eta_{n}^+ = \frac{1}{a_n} + \frac{1}{a_{n+1}} + \cdots,$$

$$\eta_{n}^- = -a'_{n-1} + \frac{1}{a_{n-2}} + \cdots,$$

$$N_{m+1} = \eta_{m+1}^+ - \eta_{m+1}^-,$$

$$a'_n = \frac{(1+\zeta)(4k^2 \alpha^2 + n^2)(4k^2 \alpha^2 + n^2 + \zeta)}{\lambda_k \alpha (4k^2 \alpha^2 + n^2 - 1)}.$$
We can prove Proposition 2 in the same way to Section 3.2 of [7]. Substituting (4.6) into (4.4), we have

\[ \iint_{D} \{ J(\Delta\varphi_{ss}(0), \varphi_{k,1}) + J(\Delta\varphi_{k,1}, \varphi_{ss}(0)) \} \Phi_{k,1} dxdy \equiv D_{1} + D_{2}, \]

\[ D_{1} \equiv \iint_{D} \{ J(\Delta Z_{1}, \varphi_{k,1}) + J(\Delta\varphi_{k,1}, Z_{1}) \} \Phi_{k,1} dxdy, \]

\[ D_{2} \equiv \iint_{D} \{ J(\Delta Z_{2}, \varphi_{k,1}) + J(\Delta\varphi_{k,1}, Z_{2}) \} \Phi_{k,1} dxdy. \]

As for \( D_{1} \) and \( D_{2} \), we obtain the following proposition.

**Proposition 3** For each fixed \( \zeta \geq 0 \), \( D_{1} > |D_{2}| \) holds if \( k\alpha \) close to one.

The proof is given in my current preprint [12], which is based on the previous paper (Section 4 and 5 of [7]). This proposition means that \( \mu_{ss}(0) > 0 \) holds if \( k\alpha \in (0, 1) \) is sufficiently close to one. Thus, Theorem 1 is proved.

**References**


