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<thead>
<tr>
<th>項目</th>
<th>内容</th>
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</thead>
<tbody>
<tr>
<td>Title</td>
<td>BSPFA Combined with One Measurable Cardinal (Studies in Relative Consistency Proofs with Particular Emphasis on Set Theoretic Methods)</td>
</tr>
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Kyoto University
BSPFA Combined with One Measurable Cardinal

Miyamoto Tadatoshi · Nanzan University

Abstract

We consider consequences of BSPFA (Bounded Semi-Proper Forcing Axiom) combined with an existence of a measurable cardinal. The large cardinal assures existences of relevant semiproper preorders via Chang’s Conjecture-type arguments.

Introduction

In [T], a new combinatorial principle \( \theta_{AC} \) is introduced. We recall its definition.

Definition. \( \theta_{AC} \) holds, if for every one-to-one list \( r = (r_i) \) in \( \omega^2 \) and every \( S \subseteq \omega_1 \), there exist ordinals \( \gamma > \beta > \alpha \geq \omega_1 \) and an increasing continuous decomposition \( \gamma = \bigcup \{ N_\nu \mid \nu < \omega_1 \} \) of the ordinal \( \gamma \) into countable sets such that for all \( \nu < \omega_1 \), \( N_\nu \cap \omega_1 \in S \) if and only if the following holds, where \( i = \text{o.t.}(N_\nu \cap \alpha), j = \text{o.t.}(N_\nu \cap \beta) \) and \( k = \text{o.t.}(N_\nu) \),

\[ \Delta(r_i, r_j) = \max \{ \Delta(r_i, r_k), \Delta(r_j, r_k) \} \]

The notation \( \Delta(r, r') \) stands for the least \( n < \omega \) such that \( r(n) \neq r'(n) \) for \( r, r' \in \omega^2 \). We also recall.

Definition. BMM (Bounded Martin’s Maximum) holds, if for any \( A \in H^V_{\omega^2} \) and any \( \Sigma_0 \)-formula \( \varphi \), if \( \models_{P} ^{\omega^2, \omega^2} \exists y \varphi(y, A) \) in \( H^V_{\omega^2} \) holds for some preorder \( P \) which preserves every stationary subset of \( \omega_1 \), then we already have \( \exists y \varphi(y, A) \) in \( H^V_{\omega^2} \).

We may formulate a weaker forcing axiom by restricting the class of preorders to the semiproper ones.

Definition. BSPFA (Bounded Semi-Proper Forcing Axiom) holds, if for any \( A \in H^V_{\omega^2} \) and any \( \Sigma_0 \)-formula \( \varphi \), if \( \models_{P} ^{\omega^2, \omega^2} \exists y \varphi(y, A) \) in \( H^V_{\omega^2} \) holds for some preorder \( P \) which is semiproper, then we already have \( \exists y \varphi(y, A) \) in \( H^V_{\omega^2} \).

In [T], it is shown

Theorem. ([T]) (1) BMM implies \( \theta_{AC} \),

(2) \( \theta_{AC} \) implies \( 2^{\omega_1} = 2^{\omega_3} = \omega_2 \).

In this note, we consider \( \theta^*_{AC} \) which is somewhat stronger than \( \theta_{AC} \) of [T] and show

(3) If BSFPA holds and there exists a measurable cardinal, then \( \theta^*_{AC} \) holds,

(4) \( \theta^*_{AC} \) implies both \( \theta_{AC} \) and CB (Complete Bounding).

While \( \theta_{AC} \) of [T] demands existences of \( \alpha, \beta \) and \( \gamma \) with \( \omega_1 \leq \alpha < \beta < \gamma \), our \( \theta^*_{AC} \) further demands \( \alpha = \omega_1 \). The consistency strength of the assumption in (1) is not well-known. A proper class of Woodin cardinals suffices (p. 867 in [W]). However they say it is unknown whether BMM implies \( 0^* \) or not.

On the other hand, if we have a type of reflecting cardinal (which itself is very much weaker than Mahlo) and a measurable cardinal above it (and so lots of measurable must exist below it), then we get the consistency of the assumption in (3) via a revised countable support iteration (say, see [M2]).
1. Basics with The One-to-one Lists in The Cantor Space

1.1 Definition. A one-to-one list \( r = \langle r_i | i < \omega_1 \rangle \) in \( \omega^2 \) means that for all \( i < \omega_1, r_i : \omega \rightarrow 2 \) and for all \( i, j < \omega_1, \) if \( i \neq j, \) then \( r_i \neq r_j. \) In this case, we denote \( \Delta(r_i, r_j) = \min \{ n < \omega | r_i(n) \neq r_j(n) \} \). More generally, we consider a one-to-one list \( r = \langle r_i | i \in T \rangle \) on a stationary set \( T \subseteq \omega_1 \) in \( \omega^2 \). For a countable set \( X \) of ordinals, \( \text{o.t.}(X) \) denotes the order type of \( X. \) Hence \( \text{o.t.}(X) < \omega_1. \) For any ordinals \( \alpha < \beta, \) if \( \text{o.t.}(X \cap \alpha) < \text{o.t.}(X \cap \beta) < \omega_1, \) then we denote \( \Delta_X(\alpha, \beta) = \Delta(\text{o.t.}(X \cap \alpha), \text{o.t.}(X \cap \beta)). \) We usually write \( \Delta_X(\alpha, \beta) \) instead of \( \Delta_X(\alpha, \beta). \) For any ordinals \( \alpha, \beta, \) and \( \gamma, \) if \( \text{o.t.}(X \cap \alpha) < \text{o.t.}(X \cap \beta) < \text{o.t.}(X \cap \gamma) < \omega_1, \) then we denote \( \text{Max} \Delta_X(\alpha, \beta, \gamma) = \text{Max} \{ \Delta_X(\alpha, \beta), \Delta_X(\alpha, \gamma), \Delta_X(\beta, \gamma) \}. \)

1.2 Lemma. Let \( r = \langle r_i | i < \omega_1 \rangle \) be a one-to-one list in \( \omega^2. \) Then there exists \( n < \omega \) such that both \( \{ i < \omega_1 | r_i(n) = 0 \} \) and \( \{ i < \omega_1 | r_i(n) = 1 \} \) are stationary.

Proof. Suppose not. For each \( n < \omega, \) there is a club \( C_n \) and \( \epsilon_n \) such that for all \( i \in C_n, \) \( r_i(n) = \epsilon_n. \) Let \( C = \bigcap \{C_n | n < \omega \}. \) Then \( C \) is a club and for all \( i \in C \) and all \( n < \omega, \) we have \( r_i(n) = \epsilon_n. \) Hence \( \{ r_i | i \in C \} \) has one element. This is a contradiction.

\( \square \)

1.3 Lemma. Let \( r = \langle r_i | i \in T \rangle \) be a one-to-one list on a stationary set \( T \) in \( \omega^2. \) Then there exist \( m < \omega \) and \( s \in m^2 \) such that both \( \{ i \in T | r_i[m = s \text{ and } r_i(m) = 0] \} \) and \( \{ i \in T | r_i[m = s \text{ and } r_i(m) = 1] \} \) are stationary.

Proof. Suppose not. For each \( m < \omega \) and \( s \in m^2, \) there exists a club \( C_{ms} \) and \( \epsilon_{ms} \) such that for all \( i \in C_{ms}, \) \( r_i(m) = \epsilon_{ms}. \) Let \( C = \bigcap \{C_{ms} | m < \omega, s \in m^2 \}. \) Then \( C \) is a club and for all \( m < \omega, \) all \( s \in m^2 \) and all \( i \in C \cap T, \) we have if \( r_i[m = s, \) then \( r_i(m) = \epsilon_{ms}. \) In particular, \( r_i(m) = \epsilon_{ms} \) for all \( i \in C \cap T. \) Hence for \( i, j \in C \cap T, \) we may show \( r_i[m = j \text{ for all } m < \omega \text{ by induction on } m. \) Hence \( \{ r_i | i \in C \cap T \} \) has one element. This is a contradiction.

\( \square \)

1.4 Lemma. Let \( r = \langle r_i | i < \omega_1 \rangle \) be a one-to-one list in \( \omega^2. \) For any stationary \( S \) and any \( n < \omega, \) there exist \( m \leq n < \omega \) and \( s \in m^2 \) such that both \( \{ i \in S | r_i[m = s \text{ and } r_i(m) = 0] \} \) and \( \{ i \in S | r_i[m = s \text{ and } r_i(m) = 1] \} \) are stationary.

Proof. Let \( S \) and \( n \) be as given. Since \( \{r_i[n | i \in S \} \) is finite, \( S \) gets partitioned into finitely many cells according to \( r_i | n_1. \) But \( S \) is stationary. Hence one of them is stationary. So there is \( t \in \omega^2 \) such that \( T = \{ i \in S | r_i[n = t] \} \) is stationary. Now may apply lemma 1.3 to a one-to-one list \( \{r_i[n,\omega] | i \in T \} \) (somewhat abusive). Hence there exist \( m \leq n < \omega \) and \( u \in m^2 \) such that both \( \{ i \in S | r_i[m = t, r_i[m, u = m_1] \} \) and \( \{ i \in S | r_i[n = t, r_i[m, m_1 = u, r_i(m) = 0] \} \) are stationary.

\( \square \)

1.5 Lemma. Let \( r = \langle r_i | i < \omega_1 \rangle \) be a one-to-one list in \( \omega^2. \) For any \( n < \omega, \) there exists a club \( C_{rn} \) such that for any \( i \in C_{rn} \) there is \( m \) with \( n \leq m < \omega \) such that both \( \{ j \in \omega_1 | r_j[m = r_i[m, r_j(m) = 0] \} \) and \( \{ j \in \omega_1 | r_j[m = r_i[m, r_j(m) = 1] \} \) are stationary.

Proof. Suppose not. For any club \( C, \) there is \( i \in C \) such that for any \( m \) with \( n \leq m < \omega, \) there is \( \eta \) such that \( \{ j \in \omega_1 | r_j[m = r_i[m, r_j(m) = \eta] \} \) is not stationary. Let \( S = \{ i \leq \omega_1 | \text{ for all } m \leq n \text{ and } \omega, \) there is \( \eta \) such that \( \{ j \in \omega_1 | r_j[m = r_i[m, r_j(m) = \eta] \} \) is not stationary \}. \) Then \( S \) is stationary. By lemma 1.4, we have \( m \) with \( n \leq m < \omega \) and \( s \in m^2 \) such that both \( S^0 = \{ i \in S | r_i[m = s, r_i(m) = 0] \} \) and \( S^1 = \{ i \in S | r_i[m = s, r_i(m) = 1] \} \) are stationary. Pick any \( i \in S^0(\neq 0). \) Then \( r_i[m = s \text{ and } i \in S. \) Hence there is \( \eta \) such that \( \{ j \in \omega_1 | r_j[m = s, r_j(m) = \eta] \} \) is not stationary. Since \( S^0 \) is stationary, we have \( \eta = 1. \) Similary, since \( S^1 \) is stationary, we have \( \eta = 0. \) This is a contradiction.

\( \square \)

1.6 Lemma. Let \( r = \langle r_i | i < \omega_1 \rangle \) be a one-to-one list in \( \omega^2. \) Then there exists a club \( C_r \) such that for any \( i \in C_r \) and any \( n < \omega, \) there is \( m \) with \( n \leq m < \omega \) such that both \( \{ j \in \omega_1 | r_j[m = r_i[m, r_j(m) = 0] \} \) and \( \{ j \in \omega_1 | r_j[m = r_i[m, r_j(m) = 1] \} \) are stationary.
Proof. Let \( C_r = \bigcap \{ C_{rn} \mid n < \omega \} \). Then this \( C_r \) works.

\[ \square \]

1.7 Lemma. Let \( r = \langle r_i \mid i < \omega_1 \rangle \) be a one-to-one list in \( \omega^2 \). Then there exists a club \( C_r \) such that for any \( i \in C_r \) and any \( n < \omega \), we have \( \{ j < \omega_1 \mid \Delta(r_i, r_j) \geq n \} \) is stationary.

Proof. The \( C_r \) above works.

\[ \square \]

1.8 Lemma. Let \( r = \langle r_i \mid i < \omega_1 \rangle \) be a one-to-one list in \( \omega^2 \). Then there exist \( n_r < \omega \) and a club \( C_r \) such that

- Both \( \{ j < \omega_1 \mid r_j(n_r) = 0 \} \) and \( \{ j < \omega_1 \mid r_j(n_r) = 1 \} \) are stationary.
- And so
- For any \( i < \omega_1 \), \( \{ j < \omega_1 \mid \Delta(r_i, r_j) \leq n_r \} \) is stationary,
  - While
- For any \( i \in C_r \) and any \( n < \omega \), \( \{ j < \omega_1 \mid \Delta(r_i, r_j) > n \} \) is stationary.

Proof. Let \( n = n_r < \omega \) be any number such that both \( \{ j < \omega_1 \mid r_j(n) = 0 \} \) and \( \{ j < \omega_1 \mid r_j(n) = 1 \} \) are stationary. Let \( C_r \) be as in above. These \( n_r \) and \( C_r \) work.

\[ \square \]

§ 2. Basics with Semiproper Preorders

2.1 Notation. Let \( \lambda \) be a regular cardinal. We write \( N < H_{\lambda} \), if the structure \( (N, \in) \) is an elementary substructure of \( (H_{\lambda}, \in) \). For \( N \) and \( M \), we denote \( M \supseteq_{\text{end}} N \), if \( M \supseteq N \) and \( M \cap \omega_1 = N \cap \omega_1 \). We write \( \langle X_i \mid i < \omega_1 \rangle \nearrow X \), if \( \langle X_i \mid i < \omega_1 \rangle \) is a sequence of continuously increasing countable subsets of \( X \) and \( \bigcup \{ X_i \mid i < \omega_1 \} = X \).

2.2 Definition. Let \( \kappa \) be a regular uncountable cardinal and \( S \subseteq [\kappa]^\omega \). We say \( S \) is semiproper, if there exists a club \( C \subseteq [H_{(2^\kappa)^+}]^\omega \) such that for any \( N < H_{(2^\kappa)^+} \) with \( N \in C \), there is a countable \( M < H_{(2^\kappa)^+} \) such that \( M \supseteq_{\text{end}} N \) and \( M \cap \kappa \in S \).

2.3 Lemma. Let \( \kappa \) be a regular uncountable cardinal, \( S, T \subseteq [\kappa]^\omega \) be semiproper and disjoint. Then for any \( B \subseteq \omega_1 \), there is a semiproper p.o. set \( P = P(S, T, B) \) such that in \( V^P \), there is \( \langle X_i \mid i < \omega_1 \rangle \nearrow \kappa \) such that for all \( i < \omega_1 \),

- If \( i \in B \), then \( X_i \in S \),
- If \( i \notin B \), then \( X_i \in T \),
  - Hence
- \( i \in B \) if and only if \( X_i \in S \).

Proof. Let \( p \in P \), if \( p = \langle X_i^p \mid i \leq \alpha^p \rangle \) such that

- \( p \) is continuously increasing and the \( X_i^p \) are countable subsets of \( \kappa \) with \( \alpha^p < \omega_1 \),
  - For \( i \leq \alpha^p \), we have
- If \( i \in B \), then \( X_i^p \in S \),
- If \( i \notin B \), then \( X_i^p \in T \).

For \( p, q \in P \), let \( q \preceq p \), if \( q \supseteq p \).
We show that this $P$ works in a series of claims.

**Claim 1.** For any $p \in P$ and any $\xi \in \kappa$, there is $X$ such that $\xi \in X$, $q = p \cup \{(\alpha^p + 1, X)\} \in P$ and $q \leq p$.

**Proof.** According to $\alpha^p + 1 \in B$ or not, we have two cases.

**Case 1.** $\alpha^p + 1 \in B$: Since $S$ is semiproper, there is a countable $M < H(2^\kappa)^+$ such that $p, \xi \in M$ and $M \cap \kappa \in S$. Let $X = M \cap \kappa$. Then this $X$ works.

**Case 2.** $\alpha^p + 1 \not\in B$: Since $T$ is semiproper, there is a countable $M < H(2^\kappa)^+$ such that $p, \xi \in M$ and $M \cap \kappa \in T$. Let $X = M \cap \kappa$. Then this $X$ works.

$\square$

**Claim 2.** For $i < \omega_1$ and $\xi \in \kappa$, $D(i, \xi) = \{q \in P \mid i \leq \alpha^q, \xi \in X^q_{\alpha^q}\}$ is open dense in $P$.

**Proof.** By induction on $i$ for all $\xi$. By claim 1, it remains to deal with limit $i$. We show this by contradiction. Suppose for any $q \leq p$, $\alpha^q < i$. It suffices to derive a contradiction. Let $(i_n \mid n < \omega)$ be increasing such that $i_0 = \alpha^p$ and $\sup\{i_n \mid n < \omega\} = i$. According to $i \in B$ or not, we have two cases.

**Case 1.** $i \in B$: Let $M < H(2^\kappa)^+$ be such that $i, p, \xi \in M$ and $M \cap \kappa \in S$. Let $(\xi_n \mid n < \omega)$ enumerate $M \cap \kappa$. By induction we have $(p_n \mid n < \omega)$ so that $p_0 = p$, $p_n \in P \cap M$, $i_n \leq \alpha^{p_{n+1}} < i$ and $\xi_n \in X^{p_{n+1}}_{\alpha^{p_{n+1}}}$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(i, M \cap \kappa)\}$. Then $q \in P$ and $q \leq p$ with $\alpha^q = i$. This is a contradiction.

**Case 2.** $i \not\in B$: Similarly to case 1, let $M < H(2^\kappa)^+$ be such that $i, p, \xi \in M$ and $M \cap \kappa \in T$. Let $(\xi_n \mid n < \omega)$ enumerate $M \cap \kappa$. By induction we have $(p_n \mid n < \omega)$ so that $p_0 = p$, $p_n \in P \cap M$, $i_n \leq \alpha^{p_{n+1}} < i$ and $\xi_n \in X^{p_{n+1}}_{\alpha^{p_{n+1}}}$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(i, M \cap \kappa)\}$. Then $q \in P$ and $q \leq p$ with $\alpha^q = i$. This is a contradiction.

$\square$

**Claim 3.** $P$ is semiproper.

**Proof.** Let $P \in N < H(2^\kappa)^+$ with $N \in C(S) \cap C(T)$, where $C(S)$ and $C(T)$ are clubs in $[H(2^\kappa)^+]^{\omega}$ associated with semiproper $S$ and $T$ respectively. Let $p \in P \cap N$. We want to find $q \leq p$ which is $(P, N)$-semi-generic. According to $N \cap \omega_1 \in B$ or not, we have two cases.

**Case 1.** $N \cap \omega_1 \in B$: Since $N \in C(S)$, we may take a countable $M < H(2^\kappa)^+$ such that $M \supseteq N$ and $M \cap \kappa \in S$. Let $(p_n \mid n < \omega)$ be a $(P, M)$-generic sequence with $p_0 = p$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa)\}$. Then by claim 2, we know that $q \in P$ and so $q \leq p$. By construction, $q$ is $(P, M)$-generic and so $(P, N)$-semi-generic.

**Case 2.** $N \cap \omega_1 \not\in B$: Similarly to case 1, take a countable $M < H(2^\kappa)^+$ such that $M \supseteq N$ and $M \cap \kappa \in T$. Let $(p_n \mid n < \omega)$ be a $(P, M)$-generic sequence with $p_0 = p$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa)\}$. Then by claim 2, we know that $q \in P$ and so $q \leq p$. By construction, $q$ is $(P, M)$-generic and so $(P, N)$-semi-generic.

$\square$

**Claim 4.** Let $G$ be any $P$-generic filter over $V$ and let $\langle X_i \mid i < \omega_1 \rangle = \bigcup G$. Then $\langle X_i \mid i < \omega_1 \rangle \upharpoonright \kappa$ and for $i < \omega_1$, we have

- If $i \in B$, then $X_i \in S$,
- If $i \not\in B$, then $X_i \in T$.

**Proof.** By construction of $P$ and claim 2. Notice that $|\kappa| = \omega_1$ holds in the extension $V[G]$.

$\square$

This completes the proof of lemma.
2.4 Lemma. Let $\kappa$ be a regular uncountable cardinal and $S \subseteq [\kappa]^\omega$ be semiproper. Then there is a semiproper p.o. set $P = P(S)$ such that in $V^P$, there is $(X_i | i < \omega_1) \not\prec \kappa$ such that for all $i < \omega_1, X_i \in S$.

Proof. The proof is entirely similar to and simpler than lemma 2.3.

\[ \Box \]

§ 3. First Use of A Measurable Cardinal and BSPFA

We prepare a lemma with a measurable cardinal which is by now well-known with stronger statements.

3.1 Lemma. Let $\kappa$ be a measurable cardinal with a normal measure $D$ on $\kappa$. Let $N$ be a countable elementary substructure of $H_{(2^\kappa)^+}$ with $D \in N$.

1. For any $\eta \in \kappa$ and any $s \in \bigcap(N \cap D)$ such that $\sup(N \cap \kappa), \eta \in s$, we may form a countable elementary substructure $M$ of $H_{(2^\kappa)^+}$ such that $N \cup \{s\} \subseteq M$ and $M \cap s = N \cap s = N \cap \kappa$.

2. There is a continuously increasing countable elementary substructures $(N_i | i < \omega_1)$ of $H_{(2^\kappa)^+}$ such that $N_0 = N$ and $(o.t.(N_i \cap \kappa) | i < \omega_1)$ is a strictly increasing continuous sequence of countable ordinals.

3. For any stationary $S \subseteq \omega_1$, there is a countable elementary substructure $M$ of $H_{(2^\kappa)^+}$ such that $N \subseteq_{\text{end}} M$ and $o.t.(M \cap \kappa) \in S$.

Proof. For (1): Let $M = \{f(s) : f \in N\}$. Then this $M$ works.

For (2): Construct $(N_i | i < \omega_1)$ by recursion on $i$. At the successor stages, apply (1). At the limit stages, just take a union.

For (3): Immediate by (2).

\[ \Box \]

3.2 Lemma. Let $\kappa$ be a measurable cardinal and $r = \langle r_i | i < \omega_1 \rangle$ be a one-to-one list in $\omega^2$. For any countable $N < H_{(2^\kappa)^+}$ with $r, \kappa \in N$ and any $n < \omega$, there exists a countable $M < H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $\Delta_M(\omega_1, \kappa) \geq n$. Namely, $S(r, \kappa, n) = \{X \subseteq [\kappa]^\omega | \Delta_X(\omega_1, \kappa) \geq n\}$ is semiproper.

Proof. Since $r \in N$, we may assume $C_r \in N$ and so $\delta = N \cap \omega_1 \in C_r$. Therefore $S = \{j < \omega_1 | \Delta(r_\delta, r_j) \geq n\}$ is stationary. Since $\kappa$ is measurable and $\kappa \in N$, we may take a countable $M < H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $j = o.t.(M \cap \kappa) \in S$. Hence $\Delta_M(\omega_1, \kappa) = \Delta(r_{o.t.(M \cap \omega_1)}, r_{o.t.(M \cap \kappa)}) = \Delta(r_\delta, r_j) \geq n$.

3.3 Lemma. Let $\kappa$ be a measurable cardinal and $r = \langle r_i | i < \omega_1 \rangle$ be a one-to-one list in $\omega^2$. For any $n < \omega$, there exists a semiproper p.o. set $P$ such that in $V^P$, there exists $(X_i | i < \omega_1) \not\prec \kappa$ such that for all $i < \omega_1, \Delta_{X_i}(\omega_1, \kappa) \geq n$.

Proof. Apply lemma 2.4 to $S(r, \kappa, n)$.

\[ \Box \]

3.4 Lemma. (BSPFA) Let a measurable cardinal exist and $r = \langle r_i | i < \omega_1 \rangle$ be a one-to-one list in $\omega^2$. For any $n < \omega$, there exists $\beta$ with $\omega_1 < \beta < \omega_2$ and $(X_i | i < \omega_1) \not\prec \beta$ such that for all $i < \omega_1, \Delta_{X_i}(\omega_1, \beta) \geq n$.

Proof. Apply BSPFA to lemma 3.3.

\[ \Box \]

§ 4. Modifications and Summary
4.1 Lemma. Let \( n < \omega, \omega_1 < \beta < \omega_2 \) and \( \langle X_i | i < \omega_1 \rangle \not
\sim \beta \) be such that for any \( i < \omega_1 \), \( \Delta_{X_i}(\omega_1, \beta) \geq n \). Then we have a continuously increasing \( \langle N_i | i < \omega_1 \rangle \) such that

- For all \( i < \omega_1 \), \( N_i \prec H_{\omega_2} \) and \( N_i \) is countable,
- \( \beta \in N_0, \bigcup \{N_i | i < \omega_1\} \supset \omega_1 \) and so \( \bigcup \{N_i | i < \omega_1\} \supset \beta \),
- For all \( i < \omega_1 \), \( \Delta_{N_i}(\omega_1, \beta) \geq n \).

**Proof.** Let \( \langle N_i | i < \omega_1 \rangle \) be any continuously increasing sequence of countable \( N_i < H_{\omega_2} \) such that \( \bigcup \{N_i | i < \omega_1\} \supset \omega_1 \) and \( \beta \in N_0 \). Then since \( \beta < \omega_2 \), we have \( \bigcup \{N_i \cap \beta | i < \omega_1\} = \beta \) and so \( C = \{i < \omega_1 : X_i = N_i \cap \beta\} \) is a club. By reenumerating \( \{N_i | i \in C\} \), we are done.

\( \square \)

4.2 Lemma. (BSPFA) Let a measurable cardinal exist and \( r = \langle r_i | i < \omega_1 \rangle \) be a one-to-one list in \( \omega^2 \). Then there exist \( n_r < \omega \), a club \( C_r, \beta_r \) with \( \omega_1 < \beta_r < \omega_2 \) and \( \langle N_r^i | i < \omega_1 \rangle \) continuously increasing such that

- Both \( \{j < \omega_1 | r_j(n_r) = 0\} \) and \( \{j < \omega_1 | r_j(n_r) = 1\} \) are stationary,
- For any \( i < \omega_1 \), \( \{j < \omega_1 | \Delta(r_i, r_j) \leq n_r\} \) is stationary,
- For any \( i \in C_r \) and any \( n < \omega \), \( \{j < \omega_1 | \Delta(r_i, r_j) > n\} \) is stationary,
- For any \( i < \omega_1 \), \( N_r^i \prec H_{\omega_2} \) and \( N_r^i \) is countable,
- \( \beta_r \in N_0^r, \bigcup \{N_r^i | i < \omega_1\} \supset \omega_1 \) and so \( \bigcup \{N_r^i \cap \beta_r | i < \omega_1\} = \beta_r \),
- For any \( i < \omega_1 \), \( \Delta_{N_r^i}(\omega_1, \beta) \geq n_r + 1 \).

**Proof.** Combine lemma 1.8, lemma 3.4 and lemma 4.1.

\( \square \)

§ 5. Second Use of The Same Measurable Cardinal and BSPFA

5.1 Definition. Let \( \theta_{AC}^\star \) denote the following statement. For any \( r \) one-to-one list in \( \omega^2 \) and any \( B \subseteq \omega_1 \), there exist \( \beta \) and \( \gamma \) with \( \omega_1 < \beta < \gamma < \omega_2 \) and \( \langle X_i | i < \omega_1 \rangle \not
\sim \gamma \) such that for any \( i < \omega_1, i \in B \) if and only if \( \Delta_{X_i}(\omega_1, \beta) = \max \Delta_{X_i}(\omega_1, \beta, \gamma) \).

It is clear that \( \theta_{AC}^\star \) implies \( \theta_{AC} \) of [T].

5.2 Theorem. (BSPFA) If there exists a measurable cardinal, then \( \theta_{AC}^\star \) holds.

We show this in a series of lemmas.

5.3 Lemma. Let \( \kappa \) be a measurable cardinal and \( r \) be a one-to-one list in \( \omega^2 \). For any \( \beta \) with \( \omega_1 < \beta < \kappa \), any countable \( N < H_{(2^\kappa)^+} \) with \( r, \beta, \kappa \in N \), there exists a countable \( M < H_{(2^\kappa)^+} \) such that \( M \supseteq_{\text{end}} N \) and \( \Delta_M(\omega_1, \beta) = \min \Delta_M(\omega_1, \beta, \kappa) \).

**Proof.** Since \( r \in N \), we may assume \( C_r \in N \) and so \( N \cap \omega_1 \subseteq C_r \). Hence for all \( n < \omega_1 \), we have \( \{j < \omega_1 | \Delta(r_{N \cap \omega_1}, r_j) \geq n\} \) is stationary. Since \( \omega_1 < \beta \), we may calculate \( \Delta_N(\omega_1, \beta) = n \). Since \( \kappa \) is a measurable cardinal, we may choose a countable \( M < H_{(2^\kappa)^+} \) such that \( M \cap \beta = N \cap \beta \), if \( j = \text{o.t.}(M \cap \kappa) \), then \( \Delta(r_{N \cap \omega_1}, r_j) \geq n + 1 \). Since \( \Delta_M(\omega_1, \beta) = \Delta_N(\omega_1, \beta) = n < \Delta(r_{N \cap \omega_1}, r_{\text{t.o.t.}(M \cap \kappa)}) = \Delta_M(\omega_1, \kappa) \), we have \( \Delta_M(\omega_1, \beta) = \min \Delta_M(\omega_1, \beta, \kappa) \).

\( \square \)

5.4 Lemma. (BSPFA) Let \( \kappa \) be a measurable cardinal and \( r \) be a one-to-one list in \( \omega^2 \). For any countable \( N < H_{(2^\kappa)^+} \) with \( r, \kappa \in N \), there exists a countable \( M < H_{(2^\kappa)^+} \) such that \( M \supseteq_{\text{end}} N \) and \( \Delta_M(\omega_1, \beta_r) = \max \Delta_M(\omega_1, \beta_r, \kappa) \).
Proof. Let $\eta = r_{N \cap \omega_{1}}(n_{r})$. Let $\bar{\eta} \in \{0, 1\}$ and $\eta \neq \bar{\eta}$. Since $\{j < \omega_{1} \mid r_{j}(n_{r}) = \bar{\eta}\}$ is stationary, we may choose a countable $M < H(2^{\omega_{1}})$ such that $M \supseteq N$ and $r_{N}(n_{r}) = \eta$. Hence $\Delta_{M}(\omega_{1}, \kappa) \leq n_{r}$. On the other hand, since we may assume $(\langle N \cap \omega_{1} \mid i < \omega_{1}\rangle \cup N \cap \omega_{1}) \subseteq N$, if $\delta = N \cap \omega_{1}$, then we have $N_{\delta} \subseteq N \cap H(\omega_{2})$. Since $\beta_{r} \in N_{\delta}$, we conclude $N_{\delta} \cap \beta_{r} = N \cap \beta_{r} = M \cap \beta_{r}$ holds. So $\Delta_{M}(\omega_{1}, \beta_{r}) = \Delta_{N}(\omega_{1}, \beta_{r}) = \Delta_{N_{\delta}}(\omega_{1}, \beta_{r}) \geq n_{r} + 1$. Therefore, $\Delta_{M}(\omega_{1}, \beta_{r}) = \Delta_{M}(\omega_{1}, \beta_{r}, \kappa)$.

\[\square\]

5.5 Lemma. (BSPFA) Let $\kappa$ be a measurable cardinal and $r$ be a one-to-one list in $\omega_{2}$. Let $S = \{X \in [\kappa]^{\omega} \mid \Delta_{X}(\omega_{1}, \beta_{r}) = \operatorname{Max}_{\omega_{1}} \Delta_{X}(\omega_{1}, \beta_{r}, \kappa)\}$ and $T = \{X \in [\kappa]^{\omega} \mid \Delta_{X}(\omega_{1}, \beta_{r}) = \operatorname{Min}_{\omega_{1}} \Delta_{X}(\omega_{1}, \beta_{r}, \kappa)\}$. Then both $S$ and $T$ are semiproper and disjoint.

Proof. By lemma 5.4 and lemma 5.3.

\[\square\]

5.6 Lemma. (BSPFA) Let $\kappa$ be a measurable cardinal. Let $r = \langle r_{i} \mid i < \omega_{1} \rangle$ be a one-to-one list in $\omega_{2}$ and $B \subseteq \omega_{1}$. Then there exists a semiproper p.o. set $P$ such that in $V^{P}$, there is $\langle Y_{i} \mid i < \omega_{1} \rangle \not\in \kappa$ such that for any $i < \omega_{1}, i \in B$ if and only if $\Delta_{Y_{i}}(\omega_{1}, \beta_{r}) = \operatorname{Max}_{\omega_{1}} \Delta_{Y_{i}}(\omega_{1}, \beta_{r}, \kappa)$.

Proof. By lemma 5.5 and lemma 2.3.

\[\square\]

Proof of theorem 5.2. Apply BSPFA to the p.o. set in lemma 5.6.

\[\square\]

§ 6. $\theta_{\mathrm{AC}}^{*}$ implies CB

6.1 Definition. CB (complete bounding) stands for the following. For any $f : \omega_{1} \to \omega_{1}$, there exist $\omega_{1} < \gamma < \omega_{2}$, a club $C$ and $(X_{i} \mid i < \omega_{1}) \not\in \gamma$ such that for all $i \in C$, $f(i) < \text{o.t.}(X_{i})$.

6.2 Theorem. $\theta_{\mathrm{AC}}^{*}$ implies CB.

Proof. We have two claims.

Claim 1. If for any one-to-one list $r$ in $\omega_{2}$, there exist $\omega_{1} < \beta < \omega_{2}$ and $(X_{i} \mid i < \omega_{1}) \not\in \beta$ such that $\Delta_{X_{i}}(\omega_{1}, \beta) > 0$, then CB holds.

Proof. Let $f : \omega_{1} \to \omega_{1}$. We may assume that for all $i < \omega_{1}, i < f(i)$ and $f$ is strictly increasing. Take a continuously increasing $(N_{i} \mid i < \omega_{1})$ such that each $N_{i}$ is countable, $N_{i} \subseteq H_{\omega_{2}}$, and $N_{i} \subseteq N_{i+1}$ and $f \in N_{0}$. Notice that $N_{i} \cap \omega_{1} < f(N_{i} \cap \omega_{1}) < N_{i+1} \cap \omega_{1}$. It is easy to construct $r$ so that $r_{N_{i} \cap \omega_{1}}(0) = 1$, for $\xi$ with $N_{i} \cap \omega_{1} < \xi \leq f(N_{i} \cap \omega_{1})$, we have $r_{\xi}(0) = 0$. By assumption get $\omega_{1} < \beta < \omega_{2}$ and $(X_{i} \mid i < \omega_{1})$ such that for all $i < \omega_{1}$, we have $\Delta_{X_{i}}(\omega_{1}, \beta) > 0$. Let $C = \{i < \omega_{1} \mid N_{i} \cap \omega_{1} = i = X_{i} \cap \omega_{1}, \omega_{1} \in X_{i}\}$. Then for $i \in C$, since $\Delta_{X_{i}}(\omega_{1}, \beta) = \Delta_{r_{i}}(r_{\text{t.o.t.}(X_{i})}) > 0$, we have $f(i) = f(N_{i} \cap \omega_{1}) < \text{o.t.}(X_{i})$.

\[\square\]

Claim 2. If $\theta_{\mathrm{AC}}^{*}$ holds, then for any one-to-one list $r$ in $\omega_{2}$, there exist $\omega_{1} < \beta < \omega_{2}$ and $(X_{i} \mid i < \omega_{1}) \not\in \beta$ such that $\Delta_{X_{i}}(\omega_{1}, \beta) > 0$.

Proof. By $\theta_{\mathrm{AC}}^{*}$ for $B = \omega_{1}$, there exist $\omega_{1} < \beta < \gamma < \omega_{2}$ and $(Y_{i} \mid i < \omega_{1}) \not\in \gamma$ such that for all $i < \omega_{1}$, we have $\Delta_{Y_{i}}(\omega_{1}, \beta) = \operatorname{Max}_{\omega_{1}} \Delta_{Y_{i}}(\omega_{1}, \beta, \gamma)$. In particular, $\Delta_{Y_{i}}(\omega_{1}, \beta) > 0$. Let $X_{i} = Y_{i} \cap \beta$. Then these $\beta$ and $(X_{i} \mid i < \omega_{1})$ work.
§ 7. Additional Observations

Now we make a few observations. We may consider to directly force our $\theta_{AC}^*$. Namely, we may add the following to [D].

**Theorem.** ([D], [M]) The following are equiconsistent.

- $\text{Con}(\text{There exists a regular cardinal } \rho \text{ such that } \{\kappa < \rho \mid \kappa \text{ is a measurable cardinal}\} \text{ is cofinal in } \rho)$.
- $\text{Con}(\theta_{AC}^*)$.
- $\text{Con}(\text{CB})$.

Hence $\theta_{AC}$ of [T] accordingly has a large cardinal upper-bound.

Next, similarly to $\text{Con}(\text{PFA}^+ + \neg \text{CB})$ (which we got from S. Todorcevic), we may show via $\omega_1$-many Cohen reals $r$,

**Theorem.** ([M]) $\text{Con}(\text{PFA}^+ \text{ and } \neg \theta_{AC})$ and so $\text{Con}(\neg \text{MA} \text{ and } \theta_{AC}^*)$.

Lastly, starting with a Souslin tree in the ground model and preserving it ([M1]), we have

**Theorem.** ([M]) $\text{Con}(\text{There exists a Souslin tree and } \theta_{AC}^*)$ and so $\text{Con}(\neg \text{MA} \text{ and } \theta_{AC}^*)$.

Among others concerning the large cardinal strength of BMM, we may ask

**Question.** Does $\theta_{AC}$ of [T] imply any large cardinal, say, CB ?

More modestly,

**Question.** Does BMM imply the Weak Chang’s Conjecture ?

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Mathematics
Nanzan University
Seirei-cho, 27, Seto-shi
489-0863, Japan
miyamoto@nanzan-u.ac.jp