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DEFINABLE FIBER BUNDLES AND AFFINENESS OF DEFINABLE $C^r$ MANIFOLDS

TOMOHIRO KAWAKAMI

1. INTRODUCTION

Semialgebraic sets and semialgebraic maps have been studied and results on them can be seen in [1]. Let $\mathcal{M}$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. Every definable category on $\mathcal{M}$ is a generalization of the semialgebraic category and the definable category on $\mathcal{R}$ coincides with the semialgebraic one [22].

Some of recent results concerning o-minimal categories are [4], [5], [6], [7], [8], [10], [11], [12], [13], [14], [15], [16], [18], [21]. Semialgebraic $G$ sets and semialgebraic $G$ vector bundles are studied in [2], [19], [20].

In this note, we are concerned with homotopy property of definable fiber bundles and affineness of definable $C^r$ manifolds. Throughout this article, the term “definable” means “definable with parameters in $\mathcal{M}$” and definable maps are assumed to be continuous.

The homotopy property for topological vector bundles is established in [9]. Its semialgebraic version, its equivariant semialgebraic version and its equivariant fiber bundle version are known in 12.7.7 [1], [2] and 2.10 [17], respectively.

We have the following as a definable fiber bundle version of this property.

**Theorem 1.1** (1.1 [15]). Let $\eta = (E, p, X, F, K)$ be a definable fiber bundle over a definable set $X$ with fiber $F$ and structure group $K$. If two definable maps $f, h : Y \rightarrow X$ between definable sets are homotopic and $Y$ is compact, then $f^*(\eta)$ and $h^*(\eta)$ are definably fiber bundle isomorphic.

Let $X$ and $Y$ be definable sets. Two definable maps $f, h : X \rightarrow Y$ are called *definably homotopic* if there exists a definable map $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$. By 1.2 [11], if two definable maps between definable sets are homotopic, then they are definably homotopic. Hence two definable maps in Theorem 1.1 are definably homotopic.

We say that $\mathcal{M}$ is *polynomially bounded* if for every function $f : \mathbb{R} \rightarrow \mathbb{R}$ definable in $\mathcal{M}$, there exist a natural number $k$ and a real number $x_0$ such that $|f(x)| \leq x^k$ for any $x > x_0$. Otherwise, $\mathcal{M}$ is called *exponential*. One of typical examples of polynomially bounded structures is $\mathcal{R}$. By a result of C. Miller [18], if $\mathcal{M}$ is exponential, then the exponential function $\mathbb{R} \rightarrow \mathbb{R}, x \mapsto e^x$ is definable. We call $\mathcal{M}$ *exponentially bounded* if for every function $h : \mathbb{R} \rightarrow \mathbb{R}$ definable in $\mathcal{M}$, there exist a natural number $l$ and a
real number $x_1$ such that $|h(x)| \leq \exp_l(x)$ for any $x > x_1$, where $\exp_l(x)$ denotes the $l$th iterate of the exponential function, e.g. $\exp_2(x) = e^{e^x}$.

**Theorem 1.2** (1.1 [10]). If $\mathcal{M}$ is exponentially bounded and $0 \leq r < \infty$, then every definable $C^r$ manifold is affine.

2. **Definable sets, definable fiber bundles and definable $C^r$ manifolds**

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, (f_i)_{i \in I}, (R_j)_{j \in J}, (c_k)_{k \in K})$ be a structure expanding $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$, where $+$ (respectively $\cdot$) : $\mathbb{R}^2$ → $\mathbb{R}$ is the additive (respectively the multiplicative) function of $\mathbb{R}$, each $f_i : \mathbb{R}^{n(i)} → \mathbb{R}$, $n(i) \in \mathbb{N} \cup \{0\}$ is a function, each $R_j \subset \mathbb{R}^{n(j)}$, $n(j) \in \mathbb{N}$ is a relation, and each $c_k$ is a constant. We say that $f$ (respectively $R$) is an $m$-place function symbol (respectively an $m$-place relation symbol) if $f : \mathbb{R}^m → \mathbb{R}$ is a function (respectively $R \subset \mathbb{R}^m$ is a relation).

A term is a finite string of symbols obtained by repeated applications of the following two rules:

1. Variables are terms.
2. If $f$ is an $m$-place function symbol of $\mathcal{M}$ and $t_1, \ldots, t_m$ are terms, then the concatenated string $f(t_1, \ldots, t_m)$ is a term.

Note that if $m = 0$, then the second rule says that constant symbols (0-place function symbols) are terms.

A formula is a finite string of symbols $s_1 \ldots s_k$, where each $s_i$ is either a variable, a function symbol, a relation symbol, one of the logical symbols $=, \neg, \lor, \land, \exists, \forall$, one of the brackets $(, )$, or comma ,. Arbitrary formulas are generated inductively by the following three rules:

1. For any two terms $t_1$ and $t_2$, $t_1 = t_2$ and $t_1 > t_2$ are formulas.
2. If $R$ is an $m$-place relation symbol and $t_1, \ldots, t_m$ are terms, then $R(t_1, \ldots, t_m)$ is a formula.
3. If $\phi$ and $\psi$ are formulas, then the negation $\neg\phi$, the disjunction $\phi \lor \psi$, and the conjunction $\phi \land \psi$ are formulas. If $\phi$ is a formula and $v$ is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

A subset $X$ of $\mathbb{R}^n$ is definable (in $\mathcal{M}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and elements $b_1, \ldots, b_m \in \mathbb{R}$ such that $X = \{(a_1, \ldots, a_n) \in \mathbb{R}^n | \phi(a_1, \ldots, a_n, b_1, \ldots, b_m) \text{ true in } \mathcal{M}\}$.

Let $K \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^m$ be definable sets. We say that a continuous map $f : K → L$ is definable (in $\mathcal{M}$) if the graph of $f$ ($\subset K \times L \subset \mathbb{R}^n \times \mathbb{R}^m$) is definable. A definable map $f : K → L$ is called a definable homeomorphism if there exists a definable map $h : L → K$ such that $f \circ h = id$ and $h \circ f = id$.

An open interval means something of the form $(a, b), a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{\infty\}$. We call $\mathcal{M}$ $o$-minimal (order-minimal) if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals. Remark that $\mathcal{R}$ is $o$-minimal [22]. For example, $\mathcal{N} = (\mathbb{R}, +, \cdot, <, \mathbb{Z})$ is an expansion of $\mathcal{R}$ but not $o$-minimal because a definable subset $\mathbb{Z}$ of $\mathbb{R}$ in $\mathcal{N}$ is not a finite union of points and open intervals.

Notice that one can consider a definable category in a structure which is not $o$-minimal. But this category does not have satisfactory properties.
Let \( U \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \) be definable open sets and \( 0 < r \leq \omega \). A \( C^r \) map \( f : U \to V \) is called a \textit{definable \( C^r \) map} if it is definable.

Let \( A \subset \mathbb{R}^n \) be a definable set and \( 0 < r \leq \omega \). A definable map \( f : A \to \mathbb{R}^m \) is a \textit{definable \( C^r \) map} if there exist a definable open set \( U \subset \mathbb{R}^n \) and a definable \( C^r \) map \( F : U \to \mathbb{R}^m \) such that \( A \subset U \) and \( f = F|A \).

The following theorem states some of useful properties of definable sets and definable maps.

**Theorem 2.1.** (1) [Definable \( C^r \) cell decomposition (e.g. 7.3.3.2 [4])]. Suppose that
\[ 0 \leq r < \infty. \]
(a) For any definable set \( A_1, \ldots, A_k \subset \mathbb{R}^n \), there exists a decomposition of \( \mathbb{R}^n \) into definable \( C^r \) cells partitioning \( A_1, \ldots, A_k \).
(b) For any definable function \( f : A \to \mathbb{R}, A \subset \mathbb{R}^n \), there exists a decomposition into definable \( C^r \) cells partitioning \( A \) such that each restriction \( f|C : C \to \mathbb{R} \) is a definable \( C^r \) map for each \( C \subset A \) of the decomposition.

(2) [Definable triangulation (e.g. (8.2.9 [4])]. Let \( S \subset \mathbb{R}^n \) be a definable set and \( S_1, \ldots, S_k \) definable subsets of \( S \). Then there exist a finite simplicial complex \( K \) in \( \mathbb{R}^n \) and a definable map \( \phi : S \to \mathbb{R}^n \) such that \( \phi \) maps \( S \) and each \( S_i \) definably homeomorphically onto a union of open simplexes of \( K \). If \( S \) is compact, then we can take \( K = \phi(S) \).

(3) [Piecewise definable trivialization (e.g. 9.1.2 [4])]. Let \( X \) and \( Y \) be definable sets and \( f : X \to Y \) a definable map. Then there exist a finite partition \( \{T_i\}_{i=1}^k \) of \( Y \) into definable sets and definable homeomorphisms \( \phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(y_i) \) such that \( f|f^{-1}(T_i) = p_i \circ \phi_i \), \( (1 \leq i \leq k) \), where \( y_i \in T_i \) and \( p_i : T_i \times f^{-1}(y_i) \to T_i \) denotes the projection.

An equivariant version and an equivariant \( C^r \) version of Theorem 2.1 (3) are proved in [14].

A group \( G \) is a \textit{definable group} if \( G \) is a definable set and the group operations \( G \times G \to G \) and \( G \to G \) are definable. A subgroup of a definable group is a \textit{definable subgroup} of it if it is a definable subset of it.

Let \( G \) be a definable group. A \textit{definable set with a definable \( G \) action} is a pair \((X, \theta)\) consisting of a definable set \( X \) and a group action \( \theta : G \times X \to X \) such that \( \theta \) is a definable map. This action is not necessarily linear.

A \textit{definable space} is an object obtained by pasting finitely many definable sets together along open definable subsets, and definable maps between definable spaces are defined similarly (see Chap. 10 [4]). Definable spaces are generalizations of semialgebraic spaces in the sense of [3].

**Definition 2.2.** (1) A topological fiber bundle \( \eta = (E, p, X, F, K) \) is called a \textit{definable fiber bundle} over \( X \) with fiber \( F \) and structure group \( K \) if the following two conditions are satisfied:

(a) The total space \( E \) is a definable space, the base space \( X \) is a definable set, the structure group \( K \) is a definable group, the fiber \( F \) is a definable set with an effective definable \( K \) action, and the projection \( p : E \to X \) is a definable map.

(b) There exists a finite family of local trivializations \( \{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i \) of \( \eta \) such that each \( U_i \) is a definable open subset of \( X \), \( \{U_i\}_i \) is a finite open
covering of $X$. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where $\pi_i$ stands for the projection $U_i \times F \to F$. For any $i$ and $j$ with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations definable.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}$ and $\{V_j, \psi_j\}$, respectively. A definable map $\bar{f} : E \to E'$ is said to be a definable fiber bundle morphism if the following two conditions are satisfied:

(a) There exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \bar{f}$.

(b) For any $i, j$ such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \bar{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in $K$, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

A definable fiber bundle morphism $\bar{f} : E \to E'$ is called a definable fiber bundle isomorphism if $X = X'$, $f = id_X$ and there exists a definable fiber bundle morphism $\bar{f}' : E' \to E$ such that $f' = id_X$, $\bar{f} \circ \bar{f}' = id$, and $\bar{f}' \circ \bar{f} = id$. We say that $\eta$ is definably trivial if $\eta$ is definably fiber bundle isomorphic to the trivial bundle $(X \times F, proj, X, F, K)$, where $proj : X \times F \to X$ denotes the projection onto the first factor.

(3) A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a definable section if for any $i$, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a principal definable fiber bundle if $F = K$ and the $K$ action on $F$ is defined by the multiplication of $K$.

**Definition 2.3.** Suppose that $0 \leq r \leq \omega$.

(1) A definable subset $X$ of $\mathbb{R}^n$ is called a $d$-dimensional definable $C^r$ submanifold of $\mathbb{R}^n$ if for any $x \in X$ there exists a definable $C^r$ diffeomorphism (a definable homeomorphism if $r = 0$) $\phi_x$ from some open definable neighborhood $U_x$ of the origin in $\mathbb{R}^n$ onto some open definable neighborhood $V_x$ of $x$ in $\mathbb{R}^n$ such that $\phi_x(0) = x, \phi_x(\mathbb{R}^d \cap U_x) = X \cap V_x$. Here $\mathbb{R}^d$ denotes the subset of $\mathbb{R}^n$ those which the last $(n-d)$ components are zero.

(2) A definable $C^r$ manifold $X$ of dimension $d$ is a $C^r$ manifold with a finite system of charts $\{\phi_i : U_i \to \mathbb{R}^d\}$ such that for each $i$ and $j$, $\phi_i(U_i \cap U_j)$ is an open definable subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable $C^r$ diffeomorphism (a definable homeomorphism if $r = 0$). We call this atlas definable $C^r$. Definable $C^r$ manifolds with compatible atlases are identified.

(3) Let $X$ (respectively $Y$) be a definable $C^r$ manifold with definable $C^r$ charts $\{\phi_i : U_i \to \mathbb{R}^n\}_i$ (respectively $\{\psi_j : V_j \to \mathbb{R}^m\}_j$). A $C^r$ map $f : X \to Y$ is said to be a definable $C^r$ map if for any $i$ and $j$, $f_i(U_i \cap U_j)$ is open and definable in $\mathbb{R}^n$ and the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \to \mathbb{R}^m$ is a definable $C^r$ map.

(4) Let $X$ and $Y$ be definable $C^r$ manifolds. We say that $X$ is definably $C^r$ diffeomorphic to $Y$ (definably homeomorphic to $Y$ if $r = 0$) if one can find definable $C^r$ maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = id$ and $h \circ f = id$.

(5) A definable $C^r$ manifold is said to be affine if it is definably $C^r$ diffeomorphic (definably homeomorphic if $r = 0$) to a definable $C^r$ submanifold of some $\mathbb{R}^d$. 
3. Sketches of proofs

Theorem 1.1 is obtained from the following three results.

**Lemma 3.1** ([15]). Let $A$ be a definable set, $X_1 = \{(x_1, x_2) \in A \times [0,1] | f_1(x_1) < x_2 \leq f_2(x_1)\}, X_2 = \{(x_1, x_2) \in A \times [0,1] | f_2(x_1) \leq x_2 < f_3(x_1)\}$ and $\eta = (E,p,X,F,K)$ a definable fiber bundle over $X = X_1 \cup X_2$, where $f_1: A \to [0,1], (1 \leq i \leq 3)$, are definable functions with $f_1 < f_2 < f_3$. If $\eta|X_1$ and $\eta|X_2$ are definably trivial, then $\eta$ is definably trivial.

**Lemma 3.2** ([15]). Let $X$ be a compact definable set and $\eta = (E,p,X \times [0,1], F,K)$ a definable fiber bundle over $X \times [0,1]$. Then there exists a finite definable open covering $\{U_i\}$ of $X$ such that each $\eta|U_i \times [0,1]$ is definably trivial.

**Theorem 3.3** ([15]). Let $X$ be a compact definable set, $r : X \times [0,1] \to X \times [0,1], r(x,t) = (x,1)$ and $\eta = (E,p,X \times [0,1], F,K)$ a definable fiber bundle over $X \times [0,1]$. Then there exists a definable fiber bundle morphism $\phi : E \to E$ with $p \circ \phi = r \circ p$.

Let $e_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}$ be the function defined by $e_n(x) = \begin{cases} e^{-exp_{n-1}(1/x^2)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, where $exp_0(x) = x$. Then elementary computations show the following proposition.

**Proposition 3.4** ([10]). (1) For any polynomial function $P(x_1,\ldots,x_n)$ in n variables,

$$\lim_{x \to 0}\frac{P(\frac{1}{x}, exp_1(\frac{1}{x^2}),\ldots,exp_{n-1}(\frac{1}{x^2}))e_n(x)}{x} = 0.$$  

(2) Every $e_n$ is a $C^\infty$ function.

Since $\mathcal{M}$ is exponentially bounded, a similar proof of C.14 [7] proves the following proposition.

**Proposition 3.5** ([7], [10]). Let $A$ be a non-empty compact definable subset of $\mathbb{R}^n$ and $f,g$ two definable functions on $A$ such that $f^{-1}(0) \subset g^{-1}(0)$. If $\mathcal{M}$ is exponentially bounded, then there exist a natural number $k$ and a positive constant $c$ such that $e_k(g) \leq c|f|$ on $A$.

Theorem 1.2 is proved by using the above two propositions.

**References**


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