<table>
<thead>
<tr>
<th>Title</th>
<th>Coefficient Estimates for Certain Classes of Analytic Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Owa, Shigeyoshi; Nishiwaki, Junichi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2002), 1276: 69-74</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42304">http://hdl.handle.net/2433/42304</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Coefficient Estimates for Certain Classes of Analytic Functions

Shigeyoshi Owa and Junichi Nishiwaki

Abstract

For some real \( \alpha (\alpha > 1) \), two subclasses \( \mathcal{M}(\alpha) \) and \( \mathcal{N}(\alpha) \) of analytic functions \( f(z) \) with \( f(0) = 0 \) and \( f'(0) = 1 \) in \( \mathbb{U} \) are introduced. The object of the present paper is to discuss the coefficient estimates for functions \( f(z) \) belonging to the classes \( \mathcal{M}(\alpha) \) and \( \mathcal{N}(\alpha) \).

1 Introduction

Let \( \mathcal{A} \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \). Let \( \mathcal{M}(\alpha) \) be the subclass of \( \mathcal{A} \) consisting of functions \( f(z) \) which satisfy

\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha \quad (z \in \mathbb{U})
\]

for some \( \alpha (\alpha > 1) \). And let \( \mathcal{N}(\alpha) \) be the subclass of \( \mathcal{A} \) consisting of functions \( f(z) \) which satisfy

\[
\text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \alpha \quad (z \in \mathbb{U})
\]

for some \( \alpha (\alpha > 1) \). Then, we see that \( f(z) \in \mathcal{N}(\alpha) \) if and only if \( zf'(z) \in \mathcal{M}(\alpha) \).

Remark 1.1. For \( 1 < \alpha \leq \frac{4}{3} \), the classes \( \mathcal{M}(\alpha) \) and \( \mathcal{N}(\alpha) \) were introduced by Uralegaddi, Ganigi and Sarangi [2].

We easily see that

Example 1.1. (i) \( f(z) = z(1 - z)^{2(\alpha-1)} \in \mathcal{M}(\alpha) \).

(ii) \( g(z) = \frac{1}{2\alpha - 1} \{ 1 - (1 - z)^{2\alpha-1} \} \in \mathcal{N}(\alpha) \).

2000 Mathematics Subject Classification: Primary 30C45

Key Words and Phrases: Analytic, univalent, starlike, convex.
2 Coefficient estimates for functions

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2.1. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leq 2(\alpha - 1)$$

for some $k (0 \leq k \leq 1)$ and some $\alpha (\alpha > 1)$, then $f(z) \in \mathcal{M}(\alpha)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leq 2(\alpha - 1) \quad (1)$$

for $f(z) \in A$. It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - k \right| < 1 \quad (z \in \mathbb{U}).$$

We note that

$$\left| \frac{zf'(z)}{f(z)} - k \right| = \left| \frac{1 - k + \sum_{n=2}^{\infty} (n-k)|a_n|z^{n-1}}{1 + k - 2\alpha + \sum_{n=2}^{\infty} (n+k-2\alpha)|a_n|z^{n-1}} \right| \leq \frac{1 - k + \sum_{n=2}^{\infty} (n-k)|a_n| |z|^{n-1}}{2\alpha - 1 - k - \sum_{n=2}^{\infty} |n+k-2\alpha||a_n||z|^{n-1}}$$

$$< \frac{1 - k + \sum_{n=2}^{\infty} (n-k)|a_n|}{2\alpha - 1 - k - \sum_{n=2}^{\infty} |n+k-2\alpha||a_n|}.$$

The last expression is bounded above by 1 if

$$1 - k + \sum_{n=2}^{\infty} (n-k)|a_n| \leq 2\alpha - 1 - k - \sum_{n=2}^{\infty} |n+k-2\alpha||a_n|$$

which is equivalent to our condition

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leq 2(\alpha - 1)$$

of the theorem. This completes the proof of the theorem.
If we take $k = 1$ and some $\alpha \left( 1 < \alpha \leq \frac{3}{2} \right)$ in Theorem 2.1, then we have

**Corollary 2.1.** If $f(z) \in A$ satisfies

$$
\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \alpha - 1
$$

for some $\alpha \left( 1 < \alpha \leq \frac{3}{2} \right)$, then $f(z) \in \mathcal{M}(\alpha)$.

**Example 2.1.** The function $f(z)$ given by

$$
f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha - 1)}{n(n+1)(n-k+|n+k-2\alpha|)} z^n
$$

belongs to the class $\mathcal{M}(\alpha)$.

For the class $\mathcal{N}(\alpha)$, we have

**Theorem 2.2.** If $f(z) \in A$ satisfies

$$
\sum_{n=2}^{\infty} n(n-k+1+|n+k-2\alpha|) |a_n| \leq 2(\alpha - 1)
$$

(2)

for some $k (0 \leq k \leq 1)$ and some $\alpha (\alpha > 1)$, then $f(z)$ belongs to the class $\mathcal{N}(\alpha)$.

**Corollary 2.2.** If $f(z) \in A$ satisfies

$$
\sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq \alpha - 1
$$

for some $\alpha \left( 1 < \alpha \leq \frac{3}{2} \right)$, then $f(z) \in \mathcal{N}(\alpha)$.

**Example 2.2.** The function

$$
f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha - 1)}{n^2(n+1)(n-k+|n+k-2\alpha|)} z^n
$$

belongs to the class $\mathcal{N}(\alpha)$.

Further, denoting by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of $A$ consisting of all starlike functions of order $\alpha$, and of all convex functions of order $\alpha$, respectively, we derive
Theorem 2.3. If \( f(z) \in A \) satisfies the coefficient inequality (1) for some \( \alpha \left( 1 < \alpha \leq \frac{k+2}{2} \leq \frac{3}{2} \right) \), then \( f(z) \in S^* \left( \frac{4-3\alpha}{3-2\alpha} \right) \). If \( f(z) \in A \) satisfies the coefficient inequality (2) for some \( \alpha \left( 1 < \alpha \leq \frac{k-2}{2} \leq \frac{3}{2} \right) \), then \( f(z) \in \mathcal{K} \left( \frac{4-3\alpha}{3-2\alpha} \right) \).

Proof. For some \( \alpha \left( 1 < \alpha \leq \frac{k+2}{2} \leq \frac{3}{2} \right) \), we see that the coefficient inequality (1) implies that
\[
\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \alpha - 1.
\]
It is well-known that if \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} |a_n| \leq 1
\]
for some \( \beta \left( 0 \leq \beta < 1 \right) \), then \( f(z) \in S^* \left( \beta \right) \) by Silverman [1]. Therefore, we have to find the smallest positive \( \beta \) such that
\[
\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} |a_n| \leq \sum_{n=2}^{\infty} \frac{n - \alpha}{\alpha - 1} |a_n| \leq 1.
\]
This gives that
\[
\beta \leq \frac{(2 - \alpha)n - \alpha}{n - 2\alpha + 1}
\]
for all \( n = 2, 3, 4, \ldots \). Noting that the right hand side of the inequality (3) is increasing for \( n \), we conclude that
\[
\beta \leq \frac{4 - 3\alpha}{3 - 2\alpha},
\]
which proves that \( f(z) \in S^* \left( \frac{4 - 3\alpha}{3 - 2\alpha} \right) \). Similarly, we can show that if \( f(z) \in A \) satisfies (2), then \( f(z) \in \mathcal{K} \left( \frac{4 - 3\alpha}{3 - 2\alpha} \right) \).

Our result for the coefficient estimates of functions \( f(z) \in \mathcal{M}(\alpha) \) is contained in

Theorem 2.4. If \( f(z) \in \mathcal{M}(\alpha) \), then
\[
|a_n| \leq \frac{\Pi_{j=2}^{n} (j + 2\alpha - 4)}{(n - 1)!} \quad (n \geq 2).
\]

Proof. Let us define the function \( p(z) \) by
\[
p(z) = \frac{\alpha - z f'(z)}{f(z)}
\]
for $f(z) \in \mathcal{M}(\alpha)$. Then $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $\text{Re}(p(z)) > 0 \ (z \in \mathbb{U})$. Therefore, if we write

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then $|p_n| \leq 2 \ (n \geq 1)$. Since

$$\alpha f(z) - z f'(z) = (\alpha - 1)p(z)f(z),$$

we obtain that

$$(1-n)a_n = (\alpha - 1)(p_{n-1} + a_2 p_{n-2} + a_3 p_{n-3} + \cdots + a_{n-1} p_1).$$

If $n = 2$, then $-a_2 = (\alpha - 1)p_1$ implies that

$$|a_2| = (\alpha - 1)|p_1| \leq 2\alpha - 2.$$

Thus the coefficient estimate (4) holds true for $n = 2$. Next, suppose that the coefficient estimate

$$|a_k| \leq \frac{\Pi_{j=2}^{k}(j+2\alpha-4)}{(k-1)!}$$

is true for all $k = 2, 3, 4, \ldots, n$. Then we have that

$$-na_{n+1} = (\alpha - 1)(p_n + a_2 p_{n-1} + a_3 p_{n-2} + \cdots + a_n p_1),$$

so that

$$n|a_{n+1}| \leq (2\alpha - 2)(1 + |a_2| + |a_3| + \cdots + |a_n|)$$

$$\leq (2\alpha - 2)\left(1 + (2\alpha - 2) + \frac{(2\alpha - 2)(2\alpha - 1)}{2!} + \cdots + \frac{\Pi_{j=2}^{n}(j+2\alpha-4)}{(n-1)!}\right)$$

$$= (2\alpha - 2)\left(\frac{(2\alpha - 1)2\alpha(2\alpha + 1)\cdots(2\alpha + n - 4)}{(n-2)!} + \frac{(2\alpha - 2)(2\alpha - 1)2\alpha\cdots(2\alpha + n - 4)}{(n-1)!}\right)$$

$$= \frac{\Pi_{j=2}^{n+1}(j+2\alpha-4)}{(n-1)!}.$$

Thus, the coefficient estimate (4) holds true for the case of $k = n + 1$. Applying the mathematical induction for the coefficient estimate (4), we complete the proof of the theorem.

For the functions $f(z)$ belonging to the class $\mathcal{N}(\alpha)$, we also have

**Theorem 2.5.** If $f(z) \in \mathcal{N}(\alpha)$, then

$$|a_n| \leq \frac{\Pi_{j=2}^{n}(j+2\alpha-4)}{n!} \quad (n \geq 2).$$

**Remark 2.1.** We can not show that Theorem 2.4 and Theorem 2.5 are sharp. If we prove that Theorem 2.4 is sharp, then the sharpness of Theorem 2.5 follows.
References


Department of Mathematics

Kinki University

Higashi-Osaka, Osaka 577-8502

Japan