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Linear independence of the values of $q$-hypergeometric series

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In the present note we are interested in linear independence of the values of a certain class of $q$-hypergeometric series and its generalizations. We give a brief history on this topic in the first section, then state our results in the second and the third sections. Our results here are in [1], a joint work with K. Väänänen.

1. A brief history

Let us call here $q$-hypergeometric series the series of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{q^{-s(z)}}{\prod_{k=0}^{n-1} P(q^{-k})} z^n,$$

where $q$ is a complex number with absolute value greater than one, $s$ is a positive integer, and $P(x)$ is a polynomial with complex coefficients satisfying $P(0) \neq 0$ and $P(q^{-n}) \neq 0$ ($n = 0, 1, 2, ...$). Note that $f(z)$ represents an entire function. By defining $R(x) = x^s P(1/x)$, the series (1.1) can be expressed as

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{\prod_{k=0}^{n-1} R(q^k)}.$$

Then, under the assumption that deg $P \leq s$ (or equivalently, $R(x)$ is a polynomial), $f(z)$ satisfies the $q$-difference equation

$$\{R(D/q) - z\} f(z) = R(1/q), \quad D f(z) := f(qz).$$

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The cases $R(x) = qx$ and $R(x) = qx - 1$ correspond to the Tschakaloff function $T_q(z)$ and the $q$-exponential function $E_q(z)$, respectively.

The study of the arithmetical nature of the values of the function $T_q(z)$ goes back to Tschakaloff [10] in 1921. He proved the linear independence over the rational number field $\mathbb{Q}$ of the numbers $1$, $T_q(\alpha_j)$ ($j = 1, \ldots, m$) under a certain condition on $q \in \mathbb{Q}$, where $\alpha_j$ are nonzero rational numbers satisfying $\alpha_i/\alpha_j \neq q^n$ ($n \in \mathbb{Z}$) for any $i \neq j$, while Skolem [8] proved a similar result involving the derivatives of the function. The former result was refined in a quantitative form by Bundschuh and Shiokawa [4], and the later result by Katsurada [5]. Note that both results are valid for $q \in \mathbb{K}$ and numbers $\alpha_j \in \mathbb{K}$ with certain conditions, here and in what follows $\mathbb{K}$ denotes $\mathbb{Q}$ or an imaginary quadratic number field. Then Stihl [9] generalized the result of Bundschuh and Shiokawa to $f(z)$ having $P(x) \in \mathbb{K}[x]$ with $\deg P < s$, and proved the linear independence over $\mathbb{K}$ of the numbers

$$1, \ f(q^k \alpha_j) \ (j = 1, \ldots, m; k = 0, 1, \ldots, s - 1)$$

in quantitative form under a certain condition on $q \in \mathbb{K}$, where $\alpha_j$ are nonzero elements of $\mathbb{K}$ satisfying the same conditions as above. Since the functional equation (1.2) for $f(z)$ with $\deg P \leq s$ has the order $s$ with respect to the $q$-difference operator $D$, this result is best possible in qualitative nature. Further, Katsurada [6] put the derivatives of the function in Stihl's result to get the linear independence over $\mathbb{K}$ of the numbers

$$(1.3) \quad 1, \ f^{(i)}(q^k \alpha_j) \ (i = 0, 1, \ldots, \ell; j = 1, \ldots, m; k = 0, 1, \ldots, s - 1)$$

in quantitative form under the same conditions as Stihl's on $q$ and $\alpha_j$'s, where $\ell$ is a nonnegative integer.

We now come to the general case in which the degree of $P(x)$ is not necessarily less than $s$. In this direction Lototsky [7] in 1943 proved an irrationality result on $E_q(\alpha)$ with $q \in \mathbb{Z}$ at a rational point $\alpha$ different from $q^n$ ($n \in \mathbb{N}$). A quantitative refinement of this result with $q \in \mathbb{K}$ was obtained by Bundschuh [3]. After the work of Stihl [9], on noting that $\{R(q^n)\}$ is a linear recurrent sequence, Bézivin [2] introduced a class of entire series as follows. Let $\{A(n)\}$ be a linear recurrent sequence of the form

$$(1.4) \quad A(n) = \lambda_1 \theta_1^n + \cdots + \lambda_h \theta_h^n \quad (n = 0, 1, 2, \ldots),$$
where \( \theta_i \) are nonzero algebraic integers and \( \lambda_i \) are nonzero algebraic numbers. Assume that \( A(n) \) belong to \( K^\times \), and that

\[
|\theta_1| > |\theta_2| \geq \cdots \geq |\theta_h| \geq 1 \quad \text{and} \quad 1 = \theta_h < |\theta_{h-1}| \text{ if } |\theta_h| = 1.
\]

Then we define an entire function \( \Phi(z) \) by

\[
\Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \prod_{k=0}^{n} A(k)}.
\]

Denote by \( \tilde{G} \) the multiplicative group generated by \( \theta_1, \ldots, \theta_h \). Bezivin [2] proved the linear independence over \( K \) of the numbers

\[
1, \Phi^{(i)}(\alpha_j) \quad (i = 0, 1, \ldots, \ell; j = 1, \ldots, m),
\]

where \( \alpha_j \) are nonzero elements of \( K \) such that \( \alpha_i/\alpha_j \notin \tilde{G} \) for any \( i \neq j \), and in addition that \( \lambda_h \alpha_j \neq \tilde{G} \) \( (j = 1, \ldots, m) \) if \( \theta_h = 1 \). This result implies that, for \( f(z) \) with \( \deg P \leq s \) and an integer \( q \) in \( K \), the numbers (1.3) without powers of \( q \) are linearly independent over \( K \).

2. Generalizations of Bezivin's result

We can relax the condition (1.5) in Bezivin's result to get the following result.

**Theorem 1.** Let \( \theta_1, \ldots, \theta_h \) be nonzero algebraic integers such that

\[
|\theta_1| > 1, \quad |\theta_1| > |\theta_2| \geq \cdots \geq |\theta_h|,
\]

and that \( |\theta_h| < |\theta_{h-1}| \) if \( |\theta_h| < 1 \) and \( \theta_h = 1 \) if \( |\theta_h| = 1 \). Let \( \{A(n)\} \) be the recurrent sequence (1.4) with nonzero algebraic numbers \( \lambda_1, \ldots, \lambda_h \), and assume that \( A(n) \) belong to \( K^\times \) for all \( n \). Let \( \alpha_1, \ldots, \alpha_m \) be elements of \( K^\times \) satisfying \( \alpha_i/\alpha_j \notin \tilde{G} \) for any \( i \neq j \). If \( \theta_h = 1 \), assume in addition that \( \lambda_h \alpha_j^{-1} \notin \tilde{G} \) \( (j = 1, \ldots, m) \). Then the numbers (1.7) are linearly independent over \( K \).

We give an example of this theorem. Let \( \{F_n\} \) be the Fibonacci sequence defined by \( F_0 = F_1 = 1 \) and \( F_{n+2} = F_{n+1} + F_n \) \( (n = 0, 1, 2, \ldots) \), which is expressed as

\[
F_n = \lambda_1 \alpha^n + \lambda_2 \beta^n \quad (n = 0, 1, 2, \ldots),
\]
where \( \alpha = (1 + \sqrt{5})/2, \beta = (1 - \sqrt{5})/2, \lambda_1 = \alpha/\sqrt{5}, \lambda_2 = -\beta/\sqrt{5}. \) Since \( \beta = -\alpha^{-1}, \) the multiplicative group generated by \( \alpha^\nu \) and \( \beta^\nu \) with a positive integer \( \nu \) is \( \langle -1 \rangle \times \langle \alpha^\nu \rangle \) or \( \langle \alpha^\nu \rangle \) according as \( \nu \) is odd or even. Hence the numbers

\[
1, \sum_{n=i}^{\infty} \frac{n(n-1)\cdots(n-i+1)\alpha_j^{n-i}}{F_0 F_{\nu} \cdots F_{n\nu}} \quad (i = 0, 1, \ldots, \ell; \, j = 1, \ldots, m)
\]

are linearly independent over \( \mathbb{Q} \), if \( \nu \) is odd and \( \alpha_j \) are nonzero rational numbers having distinct absolute values, or if \( \nu \) is even and \( \alpha_j \) are nonzero distinct rational numbers.

For the next result let \( \theta_i, \lambda_i \in K \) in the above, and assume that \( \tilde{G} \) is a free abelian group. We take a free abelian group \( \hat{G} \) of finite rank satisfying \( \tilde{G} \subseteq \hat{G} \subseteq \mathbb{Q}^\times \).

Let \( r \) be the rank of \( \hat{G} \), and \( \Theta_1, \ldots, \Theta_r \) be a set of generators of \( \hat{G} \). By using these generators we can express \( \theta_i \) as

\[
\theta_i = \Theta_1^{e(i,1)} \cdots \Theta_r^{e(i,r)} \quad (i = 1, \ldots, h).
\]

Define

\[
\hat{S} = \{ \Theta_1^{\nu_1} \cdots \Theta_r^{\nu_r} \mid 0 \leq \nu_j < s_j, j = 1, \ldots, r \},
\]

where

\[
s_j = \max(0, e(1,j), \ldots, e(h,j)) - \min(0, e(1,j), \ldots, e(h,j)) \quad (j = 1, \ldots, r).
\]

Note that \( s_j \geq 1 \) for all \( j \). Then we have the following result.

**Theorem 2.** Let the notations and the assumptions be as above. Let \( \alpha_1, \ldots, \alpha_m \) be nonzero elements of \( K \) satisfying \( \alpha_i/\alpha_j \notin \hat{G} \) for any \( i \neq j \). If \( \theta_h = 1 \), assume in addition that \( \lambda_h \alpha_j^{-1} \notin \hat{G} \) \( (j = 1, \ldots, m) \). Then the numbers

\[
1, \Phi^{(i)}(\lambda \alpha_j) \quad (i = 0, 1, \ldots, \ell; \, j = 1, \ldots, m; \lambda \in \hat{S})
\]

are linearly independent over \( K \).

### 3. q-hypergeometric series

We can apply Theorem 2 for considering the values of a series generalizing the series (1.1). Let \( q_1, \ldots, q_r \) be \( r \) nonzero multiplicatively independent integers in \( K \).
with $|q_i| > 1$ for all $i$, and $\mathcal{G}$ be the multiplicative group generated by them. Let $P(x_1, ..., x_r)$ be an element of $\mathbb{K}[x_1, ..., x_r]$ satisfying

$$P(0, ..., 0) \neq 0, \quad P(q_i^{-n}, ..., q_r^{-n}) \neq 0 \quad (n = 0, 1, 2, ...).$$

Then, for positive integers $t_1, ..., t_r$, we define

$$\phi(z) = 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{r} q_i^{-t_i(n)} \prod_{k=0}^{n-1} P(q_1^{-k}, ..., q_r^{-k}) z^n.$$  \tag{3.2}

This series is a particular case of the series (1.6), and reduces to the series (1.1) when $r = 1$. We first restrict ourselves to the case $\deg_{x_i} P \leq t_i$ $(i = 1, ..., r)$.

**Theorem 3.** Let $q_i$ be as above, and $\phi(z)$ be the series (3.2) with $\deg_{x_i} P \leq t_i$ $(i = 1, ..., r)$. Let $\alpha_1, ..., \alpha_m$ be nonzero elements of $\mathbb{K}$ such that $\alpha_i/\alpha_j \not\in \mathcal{G}$ for any $i \neq j$, and assume in addition that $p_{t_1, ..., t_r} \alpha_i^{-1} \not\in \mathcal{G}$ $(i = 1, ..., m)$ if $p_{t_1, ..., t_r} \neq 0$, where $p_{t_1, ..., t_r}$ is the coefficient of $x_1^{t_1} \cdots x_r^{t_r}$ in $P(x_1, ..., x_r)$. Then the numbers

$$1, \quad \phi^{(\ell)}(\lambda \alpha_j) \quad (i = 0, 1, ..., \ell; j = 1, ..., m; \lambda \in S_1)$$  \tag{3.3}

are linearly independent over $\mathbb{K}$, where

$$S_1 = \{q_1^{k_1} \cdots q_r^{k_r} | 0 \leq k_i < t_i (i = 1, ..., r)\}.$$  

To give a result without the condition $\deg_{x_i} P \leq t_i$ $(i = 1, ..., r)$ we assume that $P(x_1, ..., x_r)$ is a product of polynomials $P_i(x_i) \in \mathbb{K}[x_i]$.

**Theorem 4.** Let $\phi(z)$ be the series (3.2) with $P(x_1, ..., x_r) = P_1(x_1) \cdots P_r(x_r)$, where $P_i(x_i) \in \mathbb{K}[x_i]$ and the condition (3.1) is satisfied. Let $\alpha_1, ..., \alpha_m$ be nonzero elements of $\mathbb{K}$ such that $\alpha_i/\alpha_j \not\in \mathcal{G}$ for any $i \neq j$, and assume in addition that $p_{t_1, t_2} \cdots p_{r,t_r} \alpha_j^{-1} \not\in \mathcal{G}$ $(i = 1, ..., m)$ if $p_{t_1, t_2} \cdots p_{r,t_r} \neq 0$, where $p_{t_1, t_2} \cdots p_{r,t_r}$ is the coefficient of $x_i^{t_i}$ in $P_i(x_i)$. Then the numbers (3.3) with $S_2$ instead of $S_1$ are linearly independent over $\mathbb{K}$, where

$$S_2 = \{q_1^{k_1} \cdots q_r^{k_r} | 0 \leq k_i < s_i (i = 1, ..., r)\}, \quad s_i = \max(t_i, \deg P_i).$$
The following is a direct consequence of Theorem 4, which generalizes Katsu- surada's result [6] in qualitative form.

Corollary. Let $q$ be an integer in $K$ with $|q| > 1$. Let $f(z)$ be the series (1.1) with $P(z) \in K[z]$ satisfying $P(0) \neq 0, P(q^{-n}) \neq 0$ ($n = 0, 1, 2, \ldots$). Let $\alpha_1, \ldots, \alpha_m$ be nonzero elements of $K$ such that $\alpha_i/\alpha_j \neq q^n$ ($n \in \mathbb{Z}$) for any $i \neq j$. Assume in addition that $p_s \alpha_j^{-1} \neq q^n$ ($n \in \mathbb{Z}, j = 1, \ldots, m$) if $p_s \neq 0$, where $p_s$ is the coefficient of $x^s$ in $P(x)$. Then the numbers (1.3) are linearly independent over $K$.

References


[10] L. Tschakaloff, Arithmetische Eigenschaften der unendlichen Reihe $\sum_{\nu=0}^{\infty} x^\nu a^{-\frac{1}{2}\nu(r+1)}$ I, Math. Ann. 80 (1921) 62–74; II, ibid. 84 (1921), 100–114.