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Kyoto University
Singular Solutions of the Briot-Bouquet Type
Partial Differential Equations

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1 Introduction

In this talk, we will study the following type of nonlinear singular first order partial differential equations:

$$t \partial_t u = F(t, x, u, \partial_x u) \quad (1.1)$$

where $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{C}_t \times \mathbb{C}_x^n$, $\partial_x u = (\partial_1 u, \ldots, \partial_n u)$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$, and $F(t, x, u, v)$ with $v = (v_1, \ldots, v_n)$ is a function defined in a polydisk $\Delta$ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_u \times \mathbb{C}_v^n$. Let us denote $\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}$.

The assumptions are as follows:

(A1) $F(t, x, u, v)$ is holomorphic in $\Delta$,
(A2) $F(0, x, 0, 0) = 0$ in $\Delta_0$,
(A3) $\frac{\partial F}{\partial v_i}(0, x, 0, 0) = 0$ in $\Delta_0$ for $i = 1, \ldots, n$.

Definition 1.1 ([2], [3]) If the equation (1.1) satisfies (A1), (A2) and (A3) we say that the equation (1.1) is of Briot-Bouquet type with respect to $t$.

Definition 1.2 ([2], [3]) Let us define

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0),$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (1.1).

Let us denote by

1. $\mathcal{R} (\mathbb{C}\backslash \{0\})$ the universal covering space of $\mathbb{C}\backslash \{0\}$,
2. $S_\theta = \{t \in \mathcal{R} (\mathbb{C}\backslash \{0\}); |\arg t| < \theta\}$,
3. $S(\epsilon(s)) = \{t \in \mathcal{R} (\mathbb{C}\backslash \{0\}); 0 < |t| < \epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on $\mathbb{R}$,
4. $D_R = \{ x \in \mathbb{C}^n; |x_i| < R \text{ for } i = 1, \ldots, n \}$.

5. $\mathcal{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\mathbb{C}^n$.

**Definition 1.3** We define the set $\tilde{O}_+$ of all functions $u(t, x)$ satisfying the following conditions:
1. $u(t, x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some $\epsilon(s)$ and $R > 0$,
2. there is an $a > 0$ such that for any $\theta > 0$ and any compact subset $K$ of $D_R$
   
   $$\max_{x \in K} |u(t, x)| = O(|t|^a) \text{ as } t \to 0 \text{ in } S_\theta.$$ 

We know some results on the equation (1.1) of Briot-Bouquet type with respect to $t$. We concern the following result. Gérard R. and Tahara H. studied in [2] the structure of holomorphic and singular solutions of the equation (1.1) and proved the following result;

**Theorem 1.4 (Gérard R. and Tahara H.)** If the equation (1.1) is of Briot-Bouquet type and $\rho(0) \not\in \mathbb{N}^* = \{1, 2, 3, \ldots\}$ then we have;

1. (Holomorphic solutions) The equation (1.1) has a unique solution $u_0(t, x)$ holomorphic near the origin of $\mathbb{C} \times \mathbb{C}^n$ satisfying $u_0(0, x) \equiv 0$.
2. (Singular solutions) Denote by $S_+$ the set of all $\tilde{O}_+$-solutions of (1.1).

$$S_+ = \begin{cases} \{u_0(t, x)\} & \text{when } \text{Re} \rho(0) \leq 0, \\
\{u_0(t, x)\} \cup \{U(\varphi); 0 \neq \varphi(x) \in \mathcal{C}\{x\}\} & \text{when } \text{Re} \rho(0) > 0,
\end{cases}$$

where $U(\varphi)$ is an $\tilde{O}_+$-solution of (1.1) having an expansion of the following form:

$$U(\varphi) = \sum_{i \geq 1} u_i(x) t^i + \sum_{i+j+k \geq 1} \varphi_{i,j,k}(x) t^{i+j+\rho(x)} (\log t)^k, \quad \varphi_{0,0,0}(x) = \varphi(x).$$

The purpose of this paper is to determine $S_+$ in the case $\rho(0) \in \mathbb{N}^*$.

The main result of this paper is;

**Theorem 1.5** If the equation (1.1) is of Briot-Bouquet type and if $\rho(0) = N \in \mathbb{N}^*$ and $\rho(x) \not\equiv \rho(0)$, then

$$S_+ = \{U(\varphi); \varphi(x) \in \mathcal{C}\{x\}\},$$

where $U(\varphi)$ is an $\tilde{O}_+$-solution of (1.1) having an expansion of the following form:

$$U(\varphi) = u_0^0(x) t + u_0^e(x) \phi_N(t, x) + \sum_{i+j+k \geq 1} u_i^j(x) t^i \Phi_N^k + \sum_{i+j+k \geq 2} \varphi_{i,j,k}(x) t^{i+j+\rho(x)} (\log t)^k, \quad \varphi_{0,0,0}(x) = \varphi(x).$$
where \( u^{0}_{N}(x) \equiv 0, w^{0}_{0,1,0}(x) = \varphi(x) \) is an arbitrary holomorphic function and the other coefficients \( u^{\beta}_{l}(x), w^{\beta}_{i,j,k}(x) \) are holomorphic functions determined by \( w^{0}_{0,1,0}(x) \) and defined in a common disk, and

\[
l = (l_{1}, \ldots, l_{n}) \in \mathbb{N}^{n}, \quad |l| = l_{1} + \cdots + l_{n}, \quad \beta = (\beta_{l} \in \mathbb{N}; \ l \in \mathbb{N}^{n}),
\]

\[
|\beta| = \sum_{|l|\geq 0} \beta_{l}, \quad |\beta|_{p} = \sum_{|l|=p} \beta_{l}, \quad [\beta] = \sum_{|l|\geq 2} (|l|-1)\beta_{l},
\]

\[
\Phi^{\beta}_{N} = \prod_{|l|\geq 0} \left( \frac{\partial_{x} \phi_{N}}{l!} \right)^{\beta_{l}}, \quad \partial^{\beta}_{x} = \partial^{l_{1}}_{x} \ldots \partial^{l_{n}}_{x}, \quad \phi_{N}(t, x) = \frac{t^{|\rho(x)|}-t}{\rho(x)-N}.
\]

The following lemma will play an important role in the proof of Theorem 1.5.

At first, we define some notations. We set for \( l \in \mathbb{N}^{n}, e_{l} = (\beta_{k}; k \in \mathbb{N}^{n}) \) with \( \beta_{l} = 1 \) and \( \beta_{k} = 0 \) for \( k \neq l \) and for \( p \in \{1, 2, \ldots, n\} \), \( e(p) = (i_{1}, \ldots, i_{n}) \) with \( i_{p} = 1 \) and \( i_{q} = 0 \) for \( q \neq p \), and define \( l^{1} < l^{0} \) is defined by \( |l^{1}| < |l^{0}| \) and \( l_{i}^{1} \leq l_{i}^{0} \) for \( i = 1, \ldots, n \).

**Lemma 1.6** Let \( \rho(x), \phi_{N} \) and \( \Phi^{\beta}_{N} \) be as in Theorem 1.5. Then we have;

1. \( \partial^{\beta}_{x} \Phi^{\beta}_{N} = \sum_{|l|\geq 0} \beta_{l} (l_{p}+1) \Phi^{\beta-e_{l}+e_{l+e(p)}}_{N} \) for \( i = 1, \ldots, n \),

2. \( t\partial^{\beta}_{x} \phi_{N} = \rho(x) \phi_{N} + t^{N} \),

3. \( t\partial^{\beta}_{x} \Phi^{\beta}_{N} = |\beta| \rho(x) \Phi^{\beta}_{N} + \beta_{0} t^{N} \Phi^{\beta-e_{0}}_{N} \).

2 Construction of formal solutions in the case \( \rho(0) = 1 \)

By [2] (Gérard-Tahara), if the equation (1.1) is of Briot-Bouquet type with respect to \( t \), then it is enough to consider the following equation:

\[
Lu = t\partial^{\beta}_{u}u - \rho(x)u = a(x)t + G_{2}(x)(t, u, \partial_{x}u)
\]

where \( \rho(x) \) and \( a(x) \) are holomorphic functions in a neighborhood of the origin, and the function \( G_{2}(x)(t, X_{0}, X_{1}, \ldots, X_{n}) \) is a holomorphic function in a neighborhood of the origin in \( C_{x}^{n} \times C_{t} \times C_{X_{0}} \times C_{X_{1}} \times \cdots \times C_{X_{n}} \) with the following expansion:

\[
G_{2}(x)(t, X_{0}, X_{1}, \ldots, X_{n}) = \sum_{p+|\alpha|\geq 2} a_{p,\alpha}(x) t^{p} \{X_{0}\}^{\alpha_{0}} \{X_{1}\}^{\alpha_{1}} \cdots \{X_{n}\}^{\alpha_{n}}
\]

and we may assume that the coefficients \( \{a_{p,\alpha}(x)\}_{p+|\alpha|\geq 2} \) are holomorphic functions on \( D_{R_{0}} \) for a sufficiently small \( R_{0} > 0 \). Let \( 0 < R < R_{0} \). We put \( A_{p,\alpha}(R) := \max_{x \in D_{R}} |\alpha|_{p+|\alpha|\geq 2} \) for \( p+|\alpha| \geq 2 \). Then for \( 0 < r < R \)

\[
\sum_{p+|\alpha|\geq 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t^{p} X_{0}^{\alpha_{0}} X_{1}^{\alpha_{1}} \cdots X_{n}^{\alpha_{n}}
\]
is convergent in a neighborhood of the origin.

In this section, we assume $\rho(0) = 1$ and $\rho(x) \not\equiv 1$ and we will construct formal solutions of the equation (2.1). In generally, we set $u(t, x) = \sum_{i=1}^{N-1} u_i(x) t^i + t^{N-1} w(t, x)$, and we consider an equation for $w(t, x)$.

**Proposition 2.1** If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, the equation (2.1) has a family of formal solutions of the form:

$$u = u_0^0(x) \phi_1 + \sum_{m \geq 2} \sum_{i+|\beta|=m \atop |\beta| \leq m-2} u_i^\beta(x) t^i \Phi_1^\beta + w_{0,1,0}^0(x) t^{\rho(x)} + \sum_{m \geq 2} \sum_{i+j+|\beta| = m \atop j \geq 1, |\beta| \leq m-2} w_{i,j,k}^\beta(x) t^{i+j \rho(x)} \{\log t\}^k \Phi_1^\beta$$

where $w_{0,1,0}^0(x)$ is an arbitrary holomorphic function and the other coefficients $u_i^\beta(x), w_{i,j,k}^\beta(x)$ are holomorphic functions determined by $w_{0,1,0}^0(x)$ and defined in a common disk.

**Remark 2.2** By the relation $|\beta| \leq m - 2$ in summations of the above formal solution, we have $\beta_l = 0$ for any $l \in \mathbb{N}^n$ with $|l| \geq m$.

We define the following two sets $U_m$ and $W_m$ for $m \geq 1$ to prove Proposition 2.1.

**Definition 2.3** We denote by $U_m$ the set of all functions $u_m$ of the following forms:

$$u_1 = u_0^0(x) t + u_0^0(x) \phi_1,$$

$$u_m = \sum_{i+|\beta|=m \atop |\beta| \leq m-2} u_i^\beta(x) t^i \Phi_1^\beta \quad \text{for } m \geq 2,$$

and denote by $W_m$ the set of all functions $w_m$ of the following forms:

$$w_1 = w_{0,1,0}^0(x) t^{\rho(x)},$$

$$w_m = \sum_{i+j+|\beta|=m \atop j \geq 1, |\beta| \leq m-2} w_{i,j,k}^\beta(x) t^{i+j \rho(x)} \{\log t\}^k \Phi_1^\beta \quad \text{for } m \geq 2$$

where $u_i^\beta(x), w_{i,j,k}^\beta(x) \in \mathbb{C}\{x\}$.

We can rewrite the formal solution (2.3) as follows:

$$u = \sum_{m \geq 1} (u_m + w_m) \quad \text{where } u_m \in U_m, w_m \in W_m.$$


Let us show important relations of $u_m$ and $w_m$ for $m \geq 2$. By Lemma 1.6, we have

\[
Lw_m = \sum_{i+j+|\beta|=m} \sum_{j \geq 1, |\beta| \leq m-2} \{ (i + (j + |\beta| - 1) \rho(x) \} w_{i,j,k}^\beta(x) t^i j+\rho(x) \{ \log t \}^k \Phi_1^\beta
\]

\[
+ \sum_{m-1} \sum_{|\alpha|=1} \sum_{\beta} \beta \rho_{\beta}(x) \log t \Phi_1^{\beta-e_{\rho+e_{\rho+1}}}
\]

We show two lemmas.

**Lemma 2.4** If $u_m \in U_m$ and $w_m \in W_m$, then $Lu_m \in U_m$ and $Lw_m \in W_m$.

**Lemma 2.5** If $u_m \in U_m$ and $w_m \in W_m$, then the following relations hold for $i,j=1, \ldots, n$

1. $a(x)U_m \subset U_m$ and $a(x)W_m \subset W_m$ for any holomorphic function $a(x)$,
2. $tU_m$, $\phi_1 U_m \subset U_{m+1}$ and $tW_m$, $\phi_1 W_m \subset W_{m+1}$,
3. $u_m \times u_n$, $\partial_i u_m \times \partial_j u_n$, $\partial_i u_m \times u_n \in U_{m+n}$,
4. $u_m \times w_n$, $\partial_i u_m \times \partial_j w_n$, $\partial_i w_m \times u_n \in W_{m+n}$,
5. $u_m \times w_n$, $\partial_i u_m \times w_n$, $u_m \times \partial_j w_n$, $\partial_i u_m \times \partial_j w_n \in W_{m+n}$.

Let us show that $u_m$ and $w_m$ are determined inductively on $m \geq 1$. By substituting $\sum_{m \geq 1} (u_m + w_m)$ into (2.1), we have

\[
(1 - \rho(x))u_1^0(x) + u_0^e(x) = a(x),
\]

and for $m \geq 2$

\[
Lu_m = \sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_0,h_0} \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_j,h_j},
\]

\[
Lw_m = \sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} (u_{m_0,h_0} + w_{m_0,h_0}) \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j (u_{m_j,h_j} + w_{m_j,h_j})
\]

- \[
\sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_0,h_0} \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_j,h_j},
\]

where $|m_n| = \sum_{i=0}^{n} m_i(\alpha_i)$ and $m_i(\alpha_i) = m_{i,1} + \cdots + m_{i,\alpha_i}$ for $i = 0, 1, \ldots, n$.

We take any holomorphic function $\varphi(x) \in C\{x\}$ and put $u_0^0(x) = \varphi(x)$, and by (2.6), we put $u_1^0(x) \equiv 0$ and $u_0^e(x) = a(x)$. 

For $m \geq 2$, let us show that $u_m$ and $w_m$ are determined by induction. By Lemma 2.5, the right side of (2.7) belongs to $U_m$ and the right side of (2.8) belongs to $W_m$. Further by $m_{j,h_j} \geq 1$, we have $m_{j,h_j} < m$ for $h_j = 1, \ldots, \alpha_j$ and $j = 0, \ldots, n$. Then for $m \geq 2$, we compare with the coefficients of $t^i \Phi_1^\beta$ and $t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta$ respectively for (2.7) and (2.8), then put

$$
\{i + (|\beta| - 1)\rho(x)\}u_i^\beta(x) + (\beta_0 + 1)u_{i-1}^\beta(x) + \sum_{|\beta| = 1}^{m-1} \sum_{0 \leq l < \rho} (\beta_0 + 1) \frac{\partial^{\beta_0-l} \rho(x)}{(\beta - l)!} u_i^{\beta + \epsilon_0 - \epsilon_1}(x) = \Phi_i^\beta((a_{p,\alpha})_{2 \leq p + |\alpha| \leq m}, \{u_i^\beta(x)\}_{i + |\beta| < m})
$$

and

$$
\{i + (j + |\beta| - 1)\rho(x)\}w_{i,j,k}^\beta(x) + (\beta_0 + 1)w_{i-1,j,k}^\beta(x) + \sum_{|\beta| = 1}^{m-1} \sum_{0 \leq l < \rho} (\beta_0 + 1) \frac{\partial^{\beta_0-l} \rho(x)}{(\beta - l)!} w_{i,j,k}^{\beta + \epsilon_0 - \epsilon_1}(x) = \Phi_i^\beta((a_{p,\alpha})_{2 \leq p + |\alpha| \leq m}, \{u_i^\beta(x)\}_{i + |\beta| < m}, \{w_{i,j,k}^\beta(x)\}_{i + j + |\beta| < m})
$$

Hence we obtain Proposition 2.1. Q.E.D.

3 Convergence of the formal solutions in the case $\rho(0) = 1$

In this section, we show that the formal solution (2.3) converges in $\tilde{O}_+$. 

**Proposition 3.1** Let $\gamma$ satisfy $0 < \gamma < 1$ and let $\lambda$ be sufficiently large. Then for any sufficiently small $r > 0$ we have the following result: 

For any $\theta > 0$ there is an $\epsilon > 0$ such that the formal solution (2.3) converges in the following region:

$$
\{(t, x) \in C_t \times C_x^n; \ |\eta(t, \lambda)t| < \epsilon, \ |\eta(t, \lambda)^2 t^{\rho(x)}| < \epsilon, \ |\eta(t, \lambda)^{\gamma}t^\gamma| < \epsilon, \ t \in S_\theta \text{ and } x \in D_r\},
$$

where $\eta(t, \lambda) = \max \{ |(\log t)/\lambda|, 1 \}$.

In this section, we put $w_{i,0,0}^\beta(x) = u_i^\beta(x)$ and $w_{i,0,k}^\beta(x) \equiv 0$ for $k \geq 1$ in the formal solution (2.3). Then the formal solution (2.3) is as follows:

$$
\Phi = w_{0,0,0}(x)\phi_1 + w_{0,1,0}(x)t^{\rho(x)} + \sum_{m \geq 2} \sum_{l \geq 0} \sum_{|\beta| = m-k} \sum_{|\beta| = l} w_{i,j,k}^\beta(x)t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta.
$$

Let us define the following set $V_m$ for (3.1).
Definition 3.2 We denote by $V_m$ the set of all the functions $v_m$ of the following forms:

$$v_m = u_m + w_m \quad \text{for} \quad u_m \in U_m \quad \text{and} \quad w_m \in W_m.$$  

We define the following estimate for the function $v_m$.

Definition 3.3 For the function (3.2), we define

$$||v_1||_{r,c,\lambda} = ||v_1||_{r,c} = \frac{||w_{0,0,0}^{e0}||_r}{c} + ||w_{0,1,0}^{0}||_r,$$

$$||v_m||_{r,c,\lambda} = \sum_{i+j+|\beta| = m} \sum_{k \leq m-2} \frac{||w_{i,j,k}^{\beta,\lambda}||_r}{c^{<\beta>}} \quad \text{for} \quad m \geq 2$$

for $c > 0$ and $\lambda > 0$, where

$$||w_{i,j,k}^{\beta,\lambda}||_r = \max_{x \in D_r} |w_{i,j,k}^{\beta,\lambda}(x)|$$

and $< \beta > = \sum_{|l| \geq 0} (|l| + 1)\beta_l$.

We will make use of

Lemma 3.4 For a holomorphic function $f(x)$ on $D_{R_0}$, we have

$$||\partial_x^\alpha f||_R \leq \frac{\alpha!}{(R_0 - R)^{|\alpha|}} ||f||_R \quad \text{for} \quad 0 < R < R_0.$$

Proof. By Cauchy's integral formula, we have the desired result. Q.E.D

Lemma 3.5 If a holomorphic function $f(x)$ on $D_R$ satisfies

$$||f||_r \leq \frac{C}{(R-r)^p} \quad \text{for} \quad 0 < r < R$$

then we have

$$||\partial_i f||_r \leq \frac{Ce(p + 1)}{(R-r)^{p+1}} \quad \text{for} \quad 0 < r < R, \quad i = 1, \ldots, n.$$ 

For the proof, see Hörmander ([1]lemma 5.1.3)

Let us show the following estimate for the function $Lu_m$.

Lemma 3.6 Let $0 < R < R_0$. Then there exists a positive constant $\sigma$ such that for $m \geq 2$, if $v_m \in V_m$ we have

$$||Lu_m||_{r,c,\lambda} \geq \frac{\sigma}{2} m ||v_m||_{r,c,\lambda} \quad \text{for} \quad 0 < r \leq R$$

for sufficiently small $c > 0$ and sufficiently large $\lambda > 0$. 

Let us estimate the function $\partial_{i} v_{m}$.

**Definition 3.7** For the function $v_{m} \in V_{m}$ we define

$$D_{p}v_{m} := \sum_{i+j+|\beta|=m} \partial_{p}w_{i,j,k}(x)t^{i+j+p(x)}\{\log t\}^{k}\Phi_{1}^{\beta}$$

for $p = 1, \ldots, n$.

**Lemma 3.8** If $v_{m} \in V_{m}$, then for $i = 1, \ldots, n$, we have

$$||\partial_{i} v_{m}||_{r,c,\lambda} \leq ||D_{i} v_{m}||_{r,c,\lambda} + c_{0}\lambda m||v_{m}||_{r,c,\lambda} + \frac{3m-2}{c}||v_{m}||_{r,c,\lambda}$$

for $0 < r \leq R$. \hfill (3.4)

Therefore by the relations (2.7), (2.8) and Lemma 3.8, we have the following lemma.

**Lemma 3.9** If $u = \sum_{m \geq 1} v_{m}$ is a formal solution of the equation (2.1) constructed in Section 2, we have the following inequality for $v_{m}$ ($m \geq 2$):

$$||Lv_{m}||_{r,c,\lambda} \leq \sum_{p+|\alpha| \geq 2} \prod_{h_{0}=1}^{\alpha_{0}} ||a_{p,\alpha}||_{r} \prod_{h_{0}=1}^{\alpha_{0}} ||v_{m_{0},h_{0}}||_{r,c,\lambda}$$

$$\times \prod_{i=1}^{n} \prod_{h_{i}=1}^{\alpha_{i}} \{ ||D_{i} v_{m_{i},h_{i}}||_{r,c,\lambda} + c_{0}\lambda m_{i,h_{i}}||v_{m_{i},h_{i}}||_{r,c,\lambda} + \frac{3m_{i,h_{i}}-2}{c}||v_{m_{i},h_{i}}||_{r,c,\lambda} \}.$$

Let us define a majorant equation to show that the formal solution (3.1) converges.

We take $A_{1}$ so that

$$\frac{||w_{0,0,0}^{0}||_{R}}{c} + ||w_{0,1,0}^{0}||_{R} \leq A_{1},$$

$$\frac{||\partial_{i} w_{0,0,0}^{0}||_{R}}{c} + ||\partial_{i} w_{0,1,0}^{0}||_{R} \leq A_{1}$$

for $i = 1, \ldots, n$.

Then we consider the following equation:

$$\frac{\sigma}{2}Y = \frac{\sigma}{2}A_{1}t_{1}$$

$$+ \frac{1}{R-r} \sum_{p+|\alpha| \geq 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}}t_{1}^{p}Y^{\alpha_{0}} \prod_{i=1}^{n} \left( eY + c_{0}\lambda Y + \frac{3}{c}Y \right)^{\alpha_{i}}. \hfill (3.5)$$
The equation (3.5) has a unique holomorphic solution \( Y = Y(t_1) \) with \( Y(0) = 0 \) at \( (Y, t_1) = (0, 0) \) by implicit function theorem. By an easy calculation, the solution \( Y = Y(t_1) \) has the following form:

\[
Y = \sum_{m \geq 1} Y_m t_1^m \quad \text{with} \quad Y_m = \frac{C_m}{(R - r)^{m-1}}
\]

where \( Y_1 = C_1 = A_1 \) and \( C_m \geq 0 \) for \( m \geq 1 \).

Then we have;

**Lemma 3.10** For \( m \geq 1 \), we have

\[
m ||v_m||_{r,c,\lambda} \leq Y_m \quad \text{for} \quad 0 < r < R. \tag{3.6}
\]

Let us show that the formal solution (3.1) converges by using (3.6) in Lemma 3.10. We rewrite \( v_m \) as follows:

\[
v_m = \sum_{i+j+|\beta|=m} \sum_{k \leq m-2} \sum_{|l| \geq 0} \frac{\omega_{i,j,k}(x)\lambda^k}{c^{<\beta>}} t^{i+j+\rho(x)} \left( \frac{\log t}{\lambda} \right)^k \Psi_1^{\beta},
\]

where

\[
\Psi_1^{\beta} = \prod_{|l| \geq 0} \left( c^{l+1} \frac{\partial_x \phi_1}{l!} \right)^{\beta_l} \tag{3.7}
\]

Firstly let us estimate (3.7). For \( ||\phi_1||_R \), we have the following lemma.

**Lemma 3.11** For any \( \gamma \) with \( 0 < \gamma < 1 \), there is an \( R > 0 \) such that

\[
||\phi_1||_R = O(\gamma) \quad \text{as} \quad t \to 0 \quad \text{in} \quad S_\theta
\]

holds for any \( \theta > 0 \).

By Lemma 3.11, there exists a positive constant \( c_1 \) such that

\[
||\phi_1||_R \leq c_1|t|^\gamma \quad \text{in} \quad S_\theta. \tag{3.8}
\]

By Lemma 3.4 and (3.8), we have

\[
||\Psi_1^{\beta}||_r \leq \prod_{|l| \geq 0} \left( c^{l+1} \frac{c_1}{(R - r)^{|l|}} |t|^\gamma \right)^{\beta_l} = \left( \frac{c}{(R - r)} \right)^{<\beta>} (c_1(R - r)|t|^\gamma)^{|\beta|} \tag{3.9}
\]

for \( 0 < r < R < R_0 \) in \( S_\theta \).

Let us estimate \( t^{i+j+\rho(x)} \left( \frac{\log t}{\lambda} \right)^k \Psi_1^{\beta} \).

We put \( \eta(t, \lambda) = \max \left\{ \left| \frac{\log t}{\lambda} \right|, 1 \right\} \), \( c_2 = \max \left\{ \frac{c}{R-r}, 1 \right\} \) and \( c_3 = c_1(R - r) \). Since
we have $|\beta| \leq m - 2 < m = i + j + |\beta|$, $< \beta > \leq 2|\beta| + |\beta| \leq i + j + 3|\beta|$ and $k \leq i + |\beta|_0 + |\beta|_1 + 2(j - 1) \leq i + |\beta| + 2j$, we obtain

$$
\left\| t^{i+j\rho(x)} \left( \frac{\log t}{\lambda} \right)^k \Psi t^\beta \right\|_r \leq \{ |c_2 \eta(t, \lambda)t| \}^i \{ \| c_2 \eta(t, \lambda)^2 t^\rho(x) \|_r \}^j \{ |(c_2)^3 c_3 \eta(t, \lambda)t^\gamma| \}^{|eta|}
$$

in $S_\theta$. For any sufficiently small $\epsilon > 0$, there exists a sufficiently small $\delta > 0$ such that for any $t \in S_\theta$ with $0 < |t| < \delta$ we have

$$
|c_2 \eta(t, \lambda)t| < \epsilon, \quad \| c_2 \eta(t, \lambda)^2 t^\rho(x) \|_r < \epsilon, \quad |(c_2)^3 c_3 \eta(t, \lambda)t^\gamma| < \epsilon.
$$

Then by Lemma 3.10, we have

$$
\| u \|_r \leq \sum_{m \geq 1} Y_m \epsilon^m \tag{3.10}
$$

for sufficiently small $|t|$ in $S_\theta$. Hence the formal solution (3.1) converges for $x \in D_r$ and sufficiently small $|t|$ in $S_\theta$. Q.E.D.

References


