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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1234: 271-274</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41526">http://hdl.handle.net/2433/41526</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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A remark on the 2D-Euler equation

In this paper we revisit the initial value problem for the 2D-Euler equation on a bounded domain. The main object is to streamline the proof of the global existence and uniqueness of a classical solution, given in the old paper [K], although there is nothing essentially new. In particular we use the vorticity $\zeta = \partial \wedge u (= \text{curl}(u))$ as a basic ingredient of the theory. However, instead of assuming that the initial velocity $a$ is $C^{1+\theta}$ as in [K], we simply assume that $\alpha = \partial \wedge a$ is $C$ and construct a unique weak solution $u(t)$ in $\hat{L}$, to be defined below. Afterwards it is shown that if $a \in C^{1+\theta}$ then $u(t) \in C^{1+\theta}$. Almost all the necessary material is in [K]; the change is only in the order of their arrangement. Naturally we follow the notation of [K] as much as possible.

As in [K], we consider a bounded domain $\Omega \subset \mathbb{R}^2$; for simplicity we assume that $\Omega$ is smooth and simply connected, and that there is no external force. (The modification necessary for a multiply connected $\Omega$ will be commented on later.) Moreover, for notational convenience we assume that $\Omega$ is closed. (If necessary we use $\Omega^o$ to denote the interior of $\Omega$.)

We denote by $|| \ ||$ the $C(\Omega)$-norm, indiscriminately for scalar or vector valued functions. $\hat{L}(\Omega; \mathbb{R}^2)$ is the set of all vector valued functions on $\Omega$ such that

$$ f \in W^{1,p}(\Omega; \mathbb{R}^2) \quad \text{for} \quad 1 < p < \infty, \quad \text{and} $$

$$ |f(x) - f(y)| \leq \text{const.} \omega(|x - y|), \quad x, y \in \Omega, $$

where $\omega(s) = s(1 + \log^+(1/s))$. The associated norm is denoted by $||f||_{q1}$.

The initial value problem for the Euler equation is given by

$$ \partial_t u + \partial.(uu) + \partial p = 0, \quad \partial . u = 0, \quad u(0) = a. \quad (1) $$

Here $uu$ is a tensor with $jk$ component $u_ju_k$; $\partial.(uu)$ is a vector with $k$ component $\partial_j(u_ju_k)$; $\partial . u = \text{div}(u) = \partial_ju_j$. (Summation convention is used throughout.)

**Theorem I.** Let $\partial \wedge a \in C(\Omega; \mathbb{R})$ and $T > 0$. Then there is a unique weak solution $\{u, p\}$ to (1) such that

$$ u \in C(I; \hat{L}(\Omega; \mathbb{R}^2)), \quad \partial p \in \quad , \quad I = [0, T]. \quad (2) $$

If in particular $\partial \wedge a \in C^\theta(\Omega; \mathbb{R})$ for some $\theta \in (0, 1)$, then $\{u, p\}$ is a classical solution with the properties

$$ u \in C(I; C^1(\Omega; \mathbb{R}^2)) \cap B(I; C^{1+\theta}(\Omega; \mathbb{R}^2)), \quad \partial_t u \in C(I; C(\Omega; \mathbb{R}^2)), \quad \partial p \in $$
where $B$ denotes the class of bounded functions.

For the proof we introduce the (scalar) vorticity

$$\zeta = \partial \wedge u = (\partial_1 u_2 - \partial_2 u_1).$$  \hspace{1cm} (4)

As is well known $\zeta$ should satisfy the vorticity equation, which is a system consisting of

$$\partial_t \zeta + \partial.(u \zeta) = 0, \quad \zeta(0) = \alpha = \partial \wedge a.$$  \hspace{1cm} (5)

Our plan is to start with a function $\varphi$ in a certain subset $S$ of $C(Q)$, where $Q = I \times \Omega$, and determine $u \in C(Q)$, which are q.L. in $x$, such that $\partial \wedge u = \varphi$. We then solve (4) for $\zeta$, which is shown to be in a certain compact subset of $S$. Furthermore, we show that the map $\varphi \mapsto \zeta$ is continuous in $C(Q)$. A fixed point of the map, which exists by the Schauder fixed point theorem, gives a solution of the vorticity equation. $u$ will then be shown to be the unique solution of (1) together with a certain gradient $\partial p$.

**Lemma 1.** For each $\varphi \in C(Q; \mathbb{R})$, there is a unique $u \in C(I; \hat{L})$ such that

$$\partial . u(t) = 0 \quad \text{and} \quad \partial \wedge u(t) = \varphi(t) \quad \text{on} \quad \Omega, \quad ||u(t)||_0 = 0 \quad \text{on} \quad b\Omega,$$

$$||u(t)||_L \leq c||\varphi(t)||, \quad t \in I, \hspace{1cm} (6)$$

where $c$ is a constant depending only on $\Omega$.

**Proof.** This follows immediately from [K, Lemma x.x]; note that $C(Q; \mathbb{R}) = C(I; C(\Omega))$.

**Lemma 2.** Let $u \in C(Q; \mathbb{R}^2)$ such that $u(t) \in \hat{L}(\Omega)$, $\partial . u(t) = 0$ on $\Omega$ and $\nu . u(t) = 0$ on $b\Omega$. Then the ordinary differential equation $dx/dt = u(t, x)$ is uniquely solvable for any initial time $s \in I$ and any initial condition $x(s) = y \in \Omega$, with the solution (characteristic function) $x = \Phi_{t,s}(y) \in \Omega$ existing for all $t \in I$. The map $\Phi : t, s, y \mapsto x$ is continuous in the three variables. For fixed $t, s$, it is a homeomorphism of $\Omega$ onto itself, satisfying the chain rule $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$.

**Proof.** The existence of the solution for all $t, s$ is due to the fact that $\partial . u = 0$ and $\nu . u = 0$ (see [K]). The uniqueness follows from the theorem of Osgood, since $1/\omega(r)$ is not integrable near $r = 0$. For the continuity properties, see e.g. [H].

**Lemma 3.** Let $u_n, n = 1, 2, \ldots$, be a sequence of functions satisfying the assumptions of Lemma 2, with the associated map $\Phi_n$. Moreover, assume that $u_n \to u$ in $C(Q; \mathbb{R}^2)$. Then $\Phi_n \to \Phi$ in $C(Q; \mathbb{R}^2)$.
Proof. This is a continuous dependence theorem for the characteristic function. Usually it is stated as continuous dependence on a auxiliary continuous parameter $\mu$ (see e.g. [H]), but there is no difference in the proof when $\mu$ is replaced by a discrete parameter $n$.

Lemma 4 The homeomorphisms $\Phi_{t,s}$ are measure preserving.

Proof. Approximate $u$ in $\hat{L}$ by $C^1$ functions, for which $\Phi$ becomes $C^1$ in all three variables and the result is classical (see e.g. [H]). The required result follows on passing to the limit using Lemma 3.

Lemma 5 $\Phi_{t,s}(y)$ is uniformly H"older continuous in the three variables for $t, s \in I, y \in \Omega$.

Proof. The result is due to the quasi-Liapashitzian property of $u$, see [K], Lemma x.x. The Hölder exponent may be very small when $T$ is large.

Lemma 6 Let $u$ be as in Lemma 2. Then the linearized vorticity equation (2) has a weak solution $\zeta$ given by

$$\zeta(t) = \alpha \circ \Phi_{0,t}, \quad t \in I.$$  \hspace{1cm} (7)

Proof. This is well known for a classical solution if $u$ and $\alpha$ were $C^1$. As it is, it requires a proof. Obviously (7) satisfies $\zeta(0) = \alpha$, since $\Phi_{0,0}$ is the identity on $\Omega$. Thus it suffices to show that for any smooth scalar function $\chi$ on $Q$, one has

$$\partial_t <\zeta, \chi> = <\zeta_t, \chi> = <\zeta, u \cdot \partial \chi>, \quad \text{where} \quad <, > \text{denotes the scalar product on } \Omega \text{ for scalar or vector valued functions.}$$ \hspace{1cm} (8)

In view of (7) and the measure preserving property of the map $\Phi_{t,s}$, (8) is equivalent to

$$\partial_t <\alpha, \chi \circ \Phi_{t,0}> = <\alpha, (u \cdot \partial \chi) \circ \Phi_{t,0}>; \quad \text{note that } \Phi_{t,0} \text{ is the inverse map of } \Phi_{0,t}. \quad \text{Here the left member equals}$$

$$<\alpha(x), \partial_t \chi(\Phi_{t,0}(x))> = <\alpha(x), \partial \chi(\Phi_{t,0}(x)) \cdot \partial_t \Phi_{t,0}(x)> = <\alpha(x), \partial \chi(\Phi_{t,0}(x)) \cdot u(t, \Phi_{t,0}(x))>$$

which is the right member of (9), q.e.d.

Remark. It appeares that Lemma 4 is nontrivial; it would be hard to prove it without the condition $\partial_t u = 0$, which implies the measure preserving property.
Lemma 7 There is $u \in C(I; \tilde{L}(\Omega; \mathbb{R}^2))$ such that $\zeta = \partial \wedge u$ is in $C(Q; \mathbb{R})$ and is a weak solution of the vorticity equation ( ).

Proof. Let $\alpha \in C(\Omega)$ be fixed. Let $S$ be the ball in $C(Q)$ with center 0 and radius $\|\alpha\|$. For each $\varphi \in S$, construct $u$ and then $\zeta$ according to Lemmas 2 and 5. Then it is obvious that $\|\zeta\| \leq \|\alpha\|$, hence $\zeta \in S$. Thus the map $F : \varphi \mapsto \zeta$ sends $S$ into itself. $F$ is continuous in the topology of $C(Q)$, as is seen from Lemmas 2,3. Moreover, the range of $F$ is compact in $C(Q)$, since $\zeta(t, x) = \alpha(\Phi_{0,t}(x))$, where $\alpha \in C(\Omega)$ is fixed and $\Phi_{0,t}(x)$ is uniformly Hölder continuous in $t, x$ by Lemma 5. It follows from Schauder's fixed point theorem that $F$ has a fixed point $\zeta$, which is a solution of the vorticity equation.