Topology of Lagrangian Submanifolds

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Y. Eliashberg gave a talk on topology of Lagrangian submanifolds at a conference held at RIMS from 9 to 12 May 2000. Here we note only a part of his talk.

The content of Sections 1 and 2, except Theorem 1.4 can be found in [1]. Theorem 1.4 is joint with L. Polterovich and is contained in [2]. Results stated in Section 3 are extracted from a joint with M. Gromov paper [3].

1 Unknotting of Lagrangian surfaces in symplectic 4-manifold

Let \((M^{2n}, \omega)\) be a symplectic manifold. An n-dimensional submanifold \(L\) is called a Lagrangian submanifold if \(\omega|_L = 0\).

Example \(M = \mathbb{R}^{2n} = \mathbb{C}^n, \omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i,\) where \((z_1, \cdots, z_n) = (x_1 + iy_1, \cdots, x_n + iy_n)\) is the standard coordinate of \(\mathbb{C}^n,\) is a symplectic manifold. In this case, a linear n-dimensional plane \(L\) is Lagrangian if and only if \(iL \perp L.\) If instead we have \(iL \cap L,\) then \(L\) is called totally real. General totally real submanifolds are defined in an obvious manner.

We will treat \(n = 2\) case of the above example. The first result we will mention is the following unknottedness theorem.

**Theorem 1.1.** Let \(\mathbb{R}^4_+ = \{y_2 \geq 0\}\) and assume that a 2-disk \(\Delta\) is embedded in \(\mathbb{R}^4_+\) as \((\Delta, \partial \Delta) \subset (\mathbb{R}^4_+, \partial \mathbb{R}^4_+)\) and \(\partial \Delta = \{|z_1| = 1, z_2 = 0\}.\) Then, if we have \(\omega|_\Delta \geq 0,\) then \(\Delta\) is unknotted, i.e. we can isotope \(\Delta\) relative to \(\partial \Delta\) to a disk in \(\partial \mathbb{R}^4_+.\)

The proof of this theorem relies on the method of filling with holomorphic discs and we quot the necessary result here. We first define the pseudo-convexity of an oriented hypersurface \(\Sigma\) of general symplectic manifold \((M^{2n}, \omega)\)
Let $J$ be an almost complex structure on $M$ tamed by $\omega$. Then, for every point $x$ on $\Sigma$, the tangent space $T_xM$ has a $J$-invariant $(2n - 2)$ dimensional subspace $T^J_x\Sigma$. $\bigcup_{x \in M}T^J_x\Sigma$ is a $(2n - 2)$ dimensional subbundle $T^J M$ of $TM$.

Since $\Sigma$ is oriented and each $T^J_x\Sigma$ has a natural orientation as a complex vector space, the quotient 1-dimensional bundle $T\Sigma/T^J\Sigma$ is also orientable, i.e. trivial. In particular, there is a trivial sub-line bundle $\mathbb{R}$ of $T\Sigma$ such that $T\Sigma = \mathbb{R} \oplus T^J\Sigma$. Choosing a non-vanishing section $\eta$ of $\mathbb{R}$ fixes a 1-form $\alpha$ on $\Sigma$ satisfying $\alpha|_{T^J \Sigma} = 0$ and $\alpha(\eta) > 0$.

**Definition 1.1.** $\Sigma$ is called $J$-convex, or pseudoconvex if the quadratic form $t \mapsto d\alpha(t, Jt)$ on $T^J \Sigma$ is positive definite.

With this preparation, we can state the following result.

**Theorem 1.2.** Let $\Omega$ be a domain in $\mathbb{R}^4$ such that $\partial \Omega$ is pseudo convex w.r.t. some almost complex structure $J$ tamed by $\omega_0$. Let $F$ be a surface with boundary embedded in $\partial \Omega$ such that $F$ has a unique complex point which is elliptic, and $J$ is integrable near that point. Moreover, assume that there is a $J$-holomorphic disc $\Delta$ with $\partial F = \partial \Delta$ and which is transversal to $\partial \Omega$ along $\partial \Delta$. Then $F \cup \Delta$ can be filled with a family of embedded, disjoint $J$-holomorphic discs $\{D_t\}$.

Now we explain the outline of the proof of the unknottedness theorem. First, we take a large sphere $S$ in $\mathbb{R}^4$ with the center on the $y_2$-axis which intersects with the $z_1$-plane along $\partial \Delta$, and let $B$ be the interior domain of $S$. We can take a disk $F$ in $S$ whose boundary coincides with $\partial \Delta$ and has a unique complex point which is elliptic, and moreover it is isotopic to a disk on $\partial \mathbb{R}^4_+$ relative to the boundary. On the otherhand, the disk $\Delta$ can be slightly deformed by a boundary fixing isotopy so that $\omega|_{\Delta} > 0$. Taking $B$ large enough, we can suppose that $\Delta$ is contained in $B$. Then, there is an almost complex structure $J$ tamed by $\omega_0$ for which $\Delta$ is $J$-holomorphic. Moreover $J$ can be chosen integrable near the elliptic point of $F$. This will allow us to apply the filling with holomorphic disc technique to the triple $(\Omega = B, F, \Delta)$, and thus will supply us with the isotopy mentioned in the theorem.

Using the same technique, we can prove the next theorem.

**Theorem 1.3.** Let $\Pi_0$ and $\Pi_1$ denote the hyperplanes $\{y_2 = 0\}$ and $\{y_2 = 1\}$, and let $L_0$ be the Lagrangian cylinder $\{|z_1| = 1, x_2 = 0, 0 \leq y_2 \leq 1\}$. Suppose $L$ is another Lagrangian cylinder between $\Pi_0$ and $\Pi_1$ having the same boundary as $L_0$. Then, $L$ is Lagrangian isotopic to $L_0$ relative to the boundary in $\mathbb{R}^4 \setminus (D_+ \cup D_- \cup R_+)$, where $D_+ = \{|z_1| \leq 1, z_2 = 0\}$, $D_- = \{|z_1| \leq 1, z_2 = 1\}$, and $R_+ = \{y_2 \geq 1, x_2 = z_1 = 0\}$.
(Outline of the proof) We again replace the plane $\Pi_0$ by a boundary $\partial \Omega$ of a large convex domain $\Omega$ such that $\partial \Omega$ intersects with the $z_1$-plane along the unit circle $C$. As before, we can take a disk $F$ whose boundary coincides with $C$ and which has a unique complex point which is elliptic. On the otherhand, we can modify the cylinder $\Delta$ by a boundary fixing isotopy, as well as gluing a disk on the top of it, so that the resulting disk $\Delta$ will have the boundary $C$, on which the symplectic form is positive. Then, as before, we can choose an almost complex structure $J$ integrable near the elliptic point of $F$, tamed by $\omega_0$, with respect to which $\Delta$ is holomorphic, and then apply the filling with holomorphic disks technique to $(F, \Delta)$. This will supply the isotopy we want.

The next is the unknottedness result for Lagrangian knots in $\mathbb{R}^4$.

**Theorem 1.4.** There is no knotted Lagrangian plane in $\mathbb{R}^4$. That is, if $\phi : \mathbb{R}^2 \to (\mathbb{R}^4, \omega_0)$ is a Lagrangian embedding which coincides with the inclusion $i : \mathbb{R}^2 \to \mathbb{C}^2$ defined by $(x, y) \mapsto (x, 0, 0, y)$ outside of a compact set, then there is a compact supported Lagrangian isotopy between $\phi$ and $i$.

(outline of the proof) This theorem is a consequence of the following two results.

**Proposition 1.** If a Lagrangian knot $L$ in $\mathbb{R}^4$ is contained in some simple hypersurface $Q$, then $L$ is Lagrangian isotopic to the flat plane.

**Proposition 2.** For every Lagrangian knot $L$ in $\mathbb{R}^4$, there is a simple hypersurface $Q$ containing it.

We first explain the word *simple hypersurface*. Let $R$ be a oriented hypersurface in $(\mathbb{R}^4, \omega_0)$. Then, the symplectic form $\omega_0$ restricted to $R$ defines an oriented 1-dimensional distribution on $R$ by $\operatorname{Ker}\omega_0$. $R$ integrates into a 1-dimensional foliation. We call this foliation *characteristic*.

**Definition 1.2.** A hypersurface $Q$ in $\mathbb{R}^4$ is called *simple* if each leaf of its characteristic foliation is diffeomorphic to $\mathbb{R}$ and outside a compact set of $Q$, each leaf coincide with a part of one of parallel straight lines of a given direction.

The proof of proposition 1 is carried out by constructing a 2-dimensional foliation $\{M_t\}_{t \in \mathbb{R}}$ on $Q$ such that each leaf is a Lagrangian diffeomorphic to $\mathbb{R}^2$, $M_0 = L$ and $M_t$ are embedded standard $\mathbb{R}^2$s for $t < -1, t > 0$. It can be done using the characteristic foliation. As for the proof of proposition 2, we need the filling with holomorphic disks technique. Namely, one first takes a 2-dimensional foliation whose leaves consist of trajectories of the
characteristics foliation which intersect at \(-\infty\) a line, parallel to a given direction. The constructed foliation is not flat at \(+\infty\), but can be flatten via an appropriate Hamiltonian isotopy. We first fix some notations. Let \((u, v, x, y)\) be the coordinate for \(\mathbb{R}^4\), \(Q_0\) be the hyperplane \(\{v = 0\}\), \(L_0\) be the standard Lagrangian plane \(\{(u, 0, 0, y)\}\) and \(\Sigma_0 = L_0 \cap C\). Let \(C = \{(x - u)^2 + y^2 \leq 1\}\) and \(K = \{(x - u)^2 + y^2 \leq 1/2\}\) be two cylinders contained in \(\mathbb{R}^3 = \{(u, x, y)\}\). There is a convex domain \(V_{\delta}\) defined by \(V_{\delta} = \{-\delta \phi(u, x, y) < v < \delta \phi(u, x, y)\}\) where \(\delta > 0\) and \(\phi(u, x, y) = 1 - (x - y)^2 - y^2\). It satisfies \(\partial V_{\delta} \supset \partial C\). Then, by a suitable dilatation, we can suppose that our Lagrangian knot \(L\) coincides with \(L_0\) outside of \(K\) and is contained in \(V_{\delta}\). We now isotope \(C \cap \{-1 \leq u \leq 1\}\) to a set like the figure below.

![Diagram](image)

We denote this map by \(\Phi\). This can be done so that the images of the disks \(\{t\} \times D^2\) are symplectic. We call the image of the discs by \(N\). Then, there is a symplectic embedding \(\chi\) from a neighbourhood of \(N\) to \(V\) such that \(\chi(\Sigma_0) = V \cap L\) and \(\chi\) is the identity outside \(K\). We can define an almost complex structure \(J\) on \(\mathbb{R}^4\) tamed by \(\omega_0\) such that the image of the discs \(\{t\} \times D^2\) by the map \(\chi \circ \Phi\) are \(J\)-holomorphic and flat near \(\partial V\) and outside of a compact set in \(\mathbb{R}^4\). Then, since \(\partial C\) is contained in a pseudo convex boundary, examining the Maslov class of the generator of the first homology group of \(\partial C\), we see that we can extend \(\chi \circ \Phi\) to the whole cylinder \(C\) in a way that images of the discs \(\{t\} \times D^2, t \in \mathbb{R}\) are \(J\)-holomorphic and for \(|t|\) larger than 1, the map on \(\{t\} \times D^2\) is the identity. If we call this map \(F\), then \(Q = (Q_0 - C \cap \{-1 \leq u \leq 1\}) \cup F(\{-1 \leq u \leq 1\})\) is the required simple hypersurface.
2 Invariants of $S^2$-knots in $\mathbb{R}^4$ via symplectic geometry

Let $f : S^2 \to \mathbb{R}^4$ be an embedding, and $\alpha := [f]$ the isotopy class of $f$. Let us denote by $D(a, b)$ the polydisc $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq a, |z_2| \leq b\}$.

We say that the class $\alpha$ admits a $(a, b)$-realization for $a > 1$, $b > 0$ if $\alpha$ can be represented by an embedded sphere $S = \Delta \cup D \subset \mathbb{R}^4$ where $D = \{|z_1| \leq 1, z_2 = b\}$ and $\Delta$ is a 2-disk satisfying the following properties: $(\Delta, \partial\Delta) \subset (\mathbb{C}^2 \setminus \text{Int}D(a, b), \partial D(a, b))$ intersects $\partial D(a, b)$ transversely along the circle $\partial\Delta = \{|z_1| = 1, z_2 = b\}$, and $\omega|_{\Delta} > 0$.

Lemma 2.1. For any isotopy class $\alpha$ of embeddings $S^2 \to \mathbb{R}^4$, there exist $a > 1$, $b > 0$ such that $\alpha$ admits a $(a, b)$-realization.

The following theorem asserts that a symplectic 2-disc cannot be knotted not only in the half-space but even in the complement of a sufficiently large polydisc.

Theorem 2.2. If $[f]$ admits a $(3, 2)$-realization, then it is trivial.
We sketch the proof of this theorem. Set the following notations:

\[ \Omega = \{ x_2 \leq \varepsilon |z_1|^2/(1-\varepsilon)^2 \} \text{ where } z = x_2 + iy_2 \]
\[ \Sigma = \partial \Omega \cap D(a,b) \]
\[ A_{c,d} = \{ |z_1| \leq c, |y_2| \leq d \} \]
\[ \Sigma_{c,d} = A_{c,d} \cap \Sigma \]
\[ G = D(a,b) \setminus (A_{1,\varepsilon} \cap \Omega) \]
\[ S = \{ y_2 = 0, |z_1| \leq 1-\varepsilon \} \cap \Sigma. \]

Deform \( \Delta \) into the following form, and denote the resulting disc by \( \tilde{\Delta} \).

\[
\begin{array}{c}
\tilde{\Delta} \\
\Sigma \\
G
\end{array}
\]

The disc \( \tilde{\Delta} \) intersects \( \Sigma \) transversely along \( \partial \tilde{\Delta} = \{ |z_1| = 1-\varepsilon, z_2 = \varepsilon \} \). We can assume that \( \omega|_{\tilde{\Delta}} > 0 \) and \( \tilde{\Delta} \) is holomorphic near \( \partial \tilde{\Delta} \) (with respect to the standard complex structure on \( \mathbb{C}^2 \)). Let us choose an almost complex structure \( J \) on \( \mathbb{R}^4 \) such that:

- \( J \) is tamed by \( \omega \).
- \( J \) is standard on \( G \), near \( \Sigma \) and at infinity.
- \( \tilde{\Delta} \) is \( J \)-holomorphic.

Then, the theorem can be deduced from the following:

**Lemma 2.3.** The pair \((S, \tilde{\Delta})\) can be filled with \( J \)-holomorphic discs.

Let \( q \in S \) be the elliptic point of \( S \), and \( \{ \Delta_t \}_t \) be a Bishop family of \( J \)-holomorphic disks developing from \( q \). To show the lemma, it is sufficient
to prove that \( \text{Int} \Delta_t \cap \Sigma_{1,\epsilon} = \emptyset \). We want to eliminate the following case.

\[
\Sigma_{2,1}
\]

Notice that no disk can be tangent to a strictly pseudoconvex hypersurface from a convex side.

Suppose that some disc \( \Delta_t \) is tangent to \( \Sigma_{2,1} \) at a point \( p \) from the concave side. Observe that for any \( t \) we have

\[
\int_{\Delta_t} \omega < \int_S \omega = \pi(1 - \epsilon)^2 \quad \text{by Stokes' theorem.}
\]

On the other hand, holomorphic curves have the following monotonicity property:

**Lemma 2.4.** Let \( C \) be a properly embedded holomorphic curve in the open ball \( B \) of radius \( r \) in \( \mathbb{C}^n \). Suppose that \( C \) contains the center of \( B \). Then \( \text{Area} \ C \geq \pi r^2 \).

We apply this lemma to \( C = \Delta_t, \ B = B_{1-\epsilon}(p) \). By assumption, \( B \cap \Delta_t \) is contained in \( G \), and \( J \) is standard on \( G \). Therefore

\[
\pi(1 - \epsilon)^2 \leq \text{Area}(\Delta_t \cap B) \leq \int_{\Delta_t} \omega.
\]

This contradicts the inequality \( \int_{\Delta_t} \omega < \pi(1 - \epsilon)^2 \).
3 Legendrian linking problem

Let $V$ be a manifold and $PT^*(V)$ the projectivized cotangent bundle, i.e., the space of all tangent hyperplanes in $T(V)$. The manifold $PT^*(V)$ has a contact structure $\eta \subset T(PT^*(V))$ such that lift of each hypersurface $W \subset V$ to $PT^*(V)$, denote by $L_W \subset PT^*(V)$, is a Legendrian submanifold for $\eta$. Moreover, let $W \subset V$ be a smooth submanifold of positive codimension. Put

$$L_W := \left\{(w, H_w) \in PT^*(V) \mid H_w \text{ is a hypersurface such that } T_w(W) \subset H_w \subset T_w(V) \right\}.$$ 

Then $L_W$ is also a Legendrian submanifold for $\eta$. Let $W_1$ and $W_2$ be submanifolds properly immersed into $V$ such that they intersect transversely. Here "properly" means "being closed as a subset in $V". Then $L_{W_1} \cap L_{W_2} = \emptyset$. Let $L_1(t)$ and $L_2(t)$ be compact supported contact isotopies of $L_{W_1}$ and $L_{W_2}$ such that $L_1(1)$ and $L_2(1)$ have disjoint projections to $V$. We denote by $\#(\L_1(t) \times_{\text{reg}} \L_2(t))$ the minimal number of crossings between all (compact supported) contact isotopies $L_1(t)$ and $L_2(t)$ which intersect transeversely and move $L_1(0)$ and $L_2(0)$ to $L_1(1)$ and $L_2(1)$.

**Theorem 3.1.** Suppose $W_1 \cap W_2$ is compact, then we have

$$\#(\L_1(t) \times_{\text{reg}} \L_2(t)) \geq \frac{1}{2} \text{ rank } H_\ast(W_1 \cap W_2),$$

where $W_1 \cap W_2$ denote the set $\{(w_1, w_2) \in W_1 \times W_2 \mid w_1 = w_2\}$.

Let $V = W \times \mathbb{R}$, $W_1 \subset W \times \mathbb{R}$, and the projection $W_1 \to W$ has non-zero degree. Here we assume $W$ and $W_1$ connected orientable manifolds of the same dimension. One can drop the orientability condition if works with coefficient $\mathbb{Z}_2$. Moreover let $W_2 \subset W$ be a compact submanifold which lies on the left of $W_1$, i.e., $W_1 \cap \{(w_2, t_2 + t) \in W \times \mathbb{R} \mid (w_2, t_2) \in W_2, t \leq 0\} = \emptyset$.

**Theorem 3.2.** If the projection of $L_2(1)$ to $V$ lies on the right of the projection $L_1(1)$, then we have

$$\#(\L_1(t) \times_{\text{reg}} \L_2(t)) \geq \text{ rank } H^\ast(W_2).$$

The proofs of these theorems rely on the generating functions and the stable Morse theory.
Postscript. In this lecture note we could note only a part of Eliashberg's talk. He mentioned many other topics on symplectic field theory (SFT), symplectic cobordisms, compactness properties, generalized Viterbo's theorem, Lagrangian skeletons, Lagrangian tori in $\mathbb{R}^4$ and so on.

References

