Spherical 5-Designs Obtained from the Unitary Group $U_{2m}(2)$

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1 Introduction

The purpose of this talk is to give an infinite series of spherical 5-designs constructed from the unitary group over the finite field of four elements. Let $G = U_{2m}(2)$ be the unitary group of dimension $2m$ over $GF(4)$, $V = GF(4)^{2m}$ the natural module of $G$. Then $G$ acts transitively on the set $\Omega$ of (maximal) totally isotropic $m$-spaces of $V$. This permutation representation (over $\mathbb{R}$) contains an irreducible representation of dimension $d = (4^m + 2)/3$. Then one can embed the set $\Omega$ into the unit sphere $S^{d-1}$ in the Euclidean space $\mathbb{R}^d$.

**Theorem 1.** $\Omega \hookrightarrow S^{d-1} \subset \mathbb{R}^d$ is a spherical 5-design.

The inner product among the vectors of $\Omega$ embedded in $\mathbb{R}^d$ can be made rational-valued, so one obtains integral lattices after a suitable normalization. Shimada [5] considered a related family of lattices, and presented in a talk in January, 2000 at RIMS.

2 Preliminaries

A spherical $t$-design ($t \in \mathbb{Z}$, $t \geq 0$) is a finite set $\Omega \subset S^{d-1}$ such that

$$\frac{\int_{S^{d-1}} f(x)dx}{\int_{S^{d-1}} 1dx} = \frac{1}{|\Omega|} \sum_{x \in \Omega} f(x)$$

for all polynomial $f \in \mathbb{R}[X_1, \ldots, X_d]$ of degree at most $t$. Equivalently,

$$\sum_{x,y \in \Omega} Q_i(\langle x, y \rangle) = 0 \quad (1 \leq i \leq t)$$ (1)
\[ Q_0(X) = 1, \quad Q_1(X) = dX, \]
\[ \frac{k+1}{d+2k}Q_{k+1}(X) = XQ_k(X) - \frac{d+k-3}{d+2k-4}Q_{k-1}(X) \]
are suitably normalized Gegenbauer polynomials. See [1, 4] for more details on spherical designs. In what follows we simply say a \( t \)-design for a spherical \( t \)-design.

Examples of spherical designs include the 196, 560 vectors of norm 4 in the Leech lattice (a 11-design), the 240 roots of the root system \( E_8 \) (a 7-design). Moreover, if \( O(d, \mathbb{R}) \supset G \) is a finite irreducible subgroup, then every \( G \)-orbit on \( S^{d-1} \) is a 2-design. Sidelnikov [6] showed that there exists a finite group \( G \subset O(2^n, \mathbb{R}) \) such that every \( G \)-orbit on \( S^{2n-1} \) is a 7-design. In general, the Molien series of \( G \) on the space of harmonic polynomials determines \( t \) for which every \( G \)-orbit on the unit sphere becomes a \( t \)-design. [1, p.102].

To see that \( \Omega \mapsto S^{d-1} \) \( (d = (4^m + 2)/3) \) is a 5-design, we shall verify the condition (1) with \( t = 5 \). We note that the values of inner products \( \langle x, y \rangle \) are known to be \((-2)^{-j}, 0 \leq j \leq m \) (see Table 6.1 (C3) of [3]), and \( \langle x, y \rangle = (-2)^{m-j} \) if and only if the dimension of the intersection of \( x \) and \( y \) is \( m - j \) (recall that \( x, y \) are \( m \)-dimensional subspaces of \( V \)). The number of pairs \( (x, y) \in \Omega^2 \) such that \( \langle x, y \rangle = (-2)^{-j} \) is given by \( |\Omega|k_j \), where

\[
k_j = \prod_{h=1}^{j} \frac{2^{2h-1}(4^{m-h+1} - 1)}{4^{h} - 1}.
\]

With these formulas at our disposal, we can verify (1) for any given values of \( m \). However, we shall employ a more general framework to prove Theorem 1.

A comment on the peculiarity of this embedding can be found in [3, Remark, p.276].

3 The Q-polynomial property for the dual polar space associated to \( U_{2m}(2) \)

As in the previous section, we let \( m \) be a fixed positive integer, and denote by \( \Omega \) the set of totally isotropic \( m \)-spaces in the natural module \( V = GF(4)^{2m} \) of
$U_{2m}(2)$. The set $\Omega$ is called the dual polar space associated to $U_{2m}(2)$, because it is a combinatorial dual of the polar space of absolute points and totally isotropic lines of the projective space $PG(V)$ with a unitary polarity. Then $U_{2m}(2)$ acts on $\Omega$, and the permutation representation (over $\mathbb{R}$) decomposes as follows:

$$R\Omega = V_{0} \perp V_{1} \perp \cdots \perp V_{m},$$

(2)

where $V_{0}$ is the trivial module. Let $E_{i} \in M_{|\Omega|}(\mathbb{R})$ be the orthogonal projection of $R\Omega$ onto $V_{i}$. If we rearrange the ordering of $V_{i}$'s if necessary, then there exists a polynomial $v_{i}^{*}(X)$ of degree $i$ ($0 \leq i \leq m$) such that

$$|\Omega|E_{i} = v_{i}^{*}(|\Omega|E_{1}) \quad (0 \leq i \leq m),$$

where, if

$$v_{i}^{*}(X) = \sum_{j=0}^{i} c_{ij}X^{j},$$

then

$$v_{i}^{*}(|\Omega|E_{1}) = \sum_{j=0}^{i} cj|\Omega|E_{1} \circ \cdots \circ E_{1},$$

where $\circ$ denotes the entry-wise product. Roughly speaking, the existence of such polynomials is refered to as the Q-polynomial property (see [2] for details). It is known that there exist $a_{i}^{*}, b_{i}^{*}, c_{i}^{*} \in \mathbb{R}$ such that

$$Xv_{i}^{*}(X) = c_{i+1}^{*}v_{i+1}^{*}(X) + a_{i}^{*}v_{i}^{*}(X) + b_{i-1}^{*}v_{i-1}^{*}(X)$$

(3)

and $\{v_{i}^{*}(X)\}$ is a system of orthogonal polynomials.

More generally, one can define a combinatorial structure called an association scheme on which the vector space of real-valued functions on the underlying set $\Omega$ can be decomposed into a direct sum like (2), and one can define Q-polynomial property for association schemes. For precise definition, we refer to [2]. The following theorem reveals a relationship between the Q-polynomial property and spherical designs. Here we denote by $E_{1}(\Omega)$ the set of unit vectors obtained by normalizing the column vectors of the matrix.
Theorem 2. Suppose that $\Omega$ is a Q-polynomial association scheme.

(i) If $a_1^* = 0$, then $E_1(\Omega)$ is a 3-design.
(ii) If moreover, $b_0^*b_1^*c_2^* + 2(b_1^*c_2^* - b_0^2 + b_0^*) = 0$, then $E_1(\Omega)$ is a 4-design.
(iii) If moreover, $a_2^* = 0$, then $E_1(\Omega)$ is a 5-design.

If $\Omega$ is the dual polar space for $U_{2m}(2)$, then all hypotheses of the theorem are satisfied, and $\Omega$ becomes a 5-design. To check this, we reproduce a more general formula for these numbers for the dual polar spaces associated with $U_{2m}(r)$, where $r$ is a prime power. They can be deduced from the formulas in [2, Section 3.5].

$$b_i^* = \frac{(r^{2m} + r)(r^{2m+2} + (-1)^i r^{i+1})}{(r + 1)(r^{2m+2} + r^{2i+1})},$$
$$c_i^* = \frac{r^{i-1}(r^i + (-1)^{i-1})(r^{2m} + r)}{(r + 1)(r^{2m} + r^{2i-1})},$$
$$a_i^* = b_0^* - b_i^* - c_i.$$

From these formulas, one checks easily that the conditions (i)–(iii) of Theorem 2 are satisfied precisely when $r = 2$.

One can find a more general formula describing these numbers for known P- and Q-polynomial association schemes [2, Section 3.5]. Thus, it is natural to consider the following problem.

**Problem.** Classify P- and Q-polynomial association scheme $\Omega$ such that $E_1(\Omega)$ is a spherical $t$-design for $t = 4, 5, 6, \ldots$.

### 4 Proof of Theorem 2

We use the orthogonality relation of the polynomials $\{v_i^*(X)\}_{i=0}^m$ given by

$$\sum_{h=0}^m k_h v_i^*(\theta_h^*) v_j^*(\theta_h^*) = 0 \quad (i \neq j),$$

where $\theta_0^* = \dim V_0 = \text{rank} E_1 = b_0^*$, and $E_1(\Omega)$ has $|\Omega|k_h$ pairs of elements with inner product $\theta_h^*/\theta_0^*$. We shall write $d$ instead of $\theta_0^*$ to simplify the notation. In view of (1), in order to prove $E_1(\Omega)$ is a $t$-design, it suffices to show

$$\sum_{h=0}^m k_h Q_i(\theta_h^*/d) = 0 \quad (1 \leq i \leq t).$$
Lemma 3. If the polynomials $Q_s(X/d)\ (1 \leq s \leq t)$ are linear combinations of the polynomials $v_1^*(X), \ldots, v_t^*(X)$, then $E_1(\Omega)$ is a $t$-design.

Proof. Since $v_0^*(X) = 1$, the orthogonality relation (4) implies

$$\sum_{h=0}^{m} k_h v_i^*(\theta_h^*) = 0 \quad (i > 0).$$

Then the condition (5) is seen to be satisfied.

It follows from the definitions that $Q_1(X/dX) = X = v_1^*(X)$, so $E_1(\Omega)$ is always a 1-design. Also, one has

$$Q_2(\frac{X}{d}) = \frac{d+2}{2d} (c_2^* v_2^*(X) + a_1^* v_1^*(X)),$$

and hence $E_1(\Omega)$ is always a 2-design.

To prove part (i) of Theorem 1, we assume $a_1^* = 0$, so that

$$XQ_2(\frac{X}{d}) = \frac{d+2}{2d} c_2^* Xv_2^*(X). \quad (6)$$

Then

$$Q_3(\frac{X}{d}) = \frac{d+4}{3} \left( \frac{X}{d} Q_2(\frac{X}{d}) - (1 - \frac{1}{d}) Q_1(\frac{X}{d}) \right)$$

$$= \frac{d+4}{3d} \left( XQ_2(\frac{X}{d}) - (d - 1)Q_1(\frac{X}{d}) \right)$$

$$= \frac{d+4}{3d} \left( \frac{d+2}{2d} c_2^* Xv_2^*(X) - (d - 1)v_1^*(X) \right)$$

$$= \frac{(d+4)(d+2)c_2^*}{6d^2} (c_3^* v_3^*(X) + a_2^* v_2^*(X))$$

$$+ \frac{(d+4)((d+2)c_2^* b_1^* - 2d(d-1))}{6d^2} v_1^*(X).$$

Thus $Q_3(X/d)$ is a linear combination of $v_1^*(X), v_2^*(X), v_3^*(X)$.

Under the assumption of (ii), we have

$$Q_3(\frac{X}{d}) = \frac{(d+4)(d+2)c_2^*}{6d^2} (c_3^* v_3^*(X) + a_2^* v_2^*(X)). \quad (7)$$
$$Q_4\left(\frac{X}{d}\right) = \frac{d+6}{4} \left(\frac{X}{d} Q_3\left(\frac{X}{d}\right) - \frac{d}{d+2} Q_2\left(\frac{X}{d}\right)\right)$$

$$= \frac{(d+6)(d+4)(d+2)c_2^*}{24d^3}(c_3^*Xv_3^*(X) + a_2^*Xv_2^*(X)) - \frac{d+6}{8}c_2^*v_2^*(X).$$

It follows from (3) that $Q_4(X/d)$ is a linear combination of $v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X)$. Under the assumption of (iii), we have

$$Q_4\left(\frac{X}{d}\right) = \frac{(d+6)(d+4)(d+2)c_2^*}{24d^3}c_3^*Xv_3^*(X) - \frac{d+6}{8}c_2^*v_2^*(X), \quad (8)$$

which is a linear combination of $v_2^*(X), v_3^*(X), v_4^*(X)$ by (3). Thus $XQ_4(X/d)$ is a linear combination of $v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X), v_5^*(X)$ by (3). Since

$$Q_5\left(\frac{X}{d}\right) = \frac{d+8}{5} \left(\frac{X}{d} Q_4\left(\frac{X}{d}\right) - \frac{d+1}{d+4} Q_3\left(\frac{X}{d}\right)\right)$$

and $Q_3(X/d)$ is a scalar multiple of $v_3^*(X)$ by (7), we see that $Q_5(X/d)$ is a linear combination of $v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X), v_5^*(X)$. This completes the proof of Theorem 2.

References


