Notes on discrete subgroups of \( PU(1,2; C) \)
with Heisenberg translations III

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In a previous paper [8] we have seen that under some conditions Parker’s theorem yields the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. In this paper we give a new stable basin region and show the same result as in [8] without the assumption on \( r \). This is a joint work with John R. Parker.

1. First we recall some definitions and notation. Let \( C \) be the field of complex numbers. Let \( V = V^{1,2}(C) \) denote the vector space \( C^3 \), together with the unitary structure defined by the Hermitian form

\[
\tilde{\Phi}(z^*, w^*) = -(z_0^* w_1^* + z_1^* w_0^*) + z_2^* w_2^*
\]

for \( z^* = (z_0^*, z_1^*, z_2^*) \), \( w^* = (w_0^*, w_1^*, w_2^*) \) in \( V \). An automorphism \( g \) of \( V \), that is a linear bijection such that \( \tilde{\Phi}(g(z^*), g(w^*)) = \tilde{\Phi}(z^*, w^*) \) for \( z^*, w^* \) in \( V \), will be called a unitary transformation. We denote the group of all unitary transformations by \( U(1,2; C) \). Let \( V_0 = \{ w^* \in V | \tilde{\Phi}(w^*, w^*) = 0 \} \) and \( V_- = \{ w^* \in V | \tilde{\Phi}(w^*, w^*) < 0 \} \). It is clear that \( V_0 \) and \( V_- \) are invariant under \( U(1,2; C) \). We denote \( U(1,2; C)/(center) \) by \( PU(1,2; C) \). Set \( V^* = V_- \cup V_0 \cup \{ 0 \} \). Let \( \pi : V^* \rightarrow \pi(V^*) \) be the projection map defined by \( \pi(w_0^*, w_1^*, w_2^*) = (w_1, w_2) \), where \( w_1 = w_0^* / w_1^* \) and \( w_2 = w_0^* / w_2^* \). We write \( \infty \) for \( (0,1,0) \). We may identify \( \pi(V_-) \) with the Siegel domain

\[
H^2 = \{ w = (w_1, w_2) \in C^2 | \ Re(w_1) > \frac{1}{2} | w_2 |^2 \}.
\]

We can regard an element of \( PU(1,2; C) \) as a transformation acting on \( H^2 \) and its boundary \( \partial H^2 \) (see [6]). Denote \( H^2 \cup \partial H^2 \) by \( \overline{H^2} \). We define a new coordinate system in \( \overline{H^2} - \{ \infty \} \). Our convention slightly differs from Basmajian-Miner [1] and Parker [8]. The \( H \) - coordinates of a point \( (w_1, w_2) \in \overline{H^2} - \{ \infty \} \) are defined by \( (k, t, w_2)_H \in (R^* \cup \{0\}) \times R \times C \) such that \( k = Re(w_1) - \frac{1}{2} | w_2 |^2 \) and \( t = Im(w_1) \). For simplicity, we write \( (t_1, w')_H \) for \( (0, t_1, w')_H \). The Cygan metric \( \rho(p, q) \) for \( p = (k_1, t_1, w)_H \) and \( q = (k_2, t_2, W')_H \) is given by

\[
\rho(p, q) = | \{ \frac{1}{2} | W' - w' |^2 + | k_2 - k_1 | \} + i \{ t_1 - t_2 + Im(w' W') \} |^\frac{1}{2}.
\]

We note that the Cygan metric \( \rho \) is a generalization of the Heisenberg metric \( \delta \) in \( \partial H^2 \).
Let $f = (a_{ij})_{1 \leq i,j \leq 3}$ be an element of $PU(1,2; \mathbb{C})$ with $f(\infty) \neq \infty$. We define the isometric sphere $I_f$ of $f$ by

$$I_f = \{ w = (w_1, w_2) \in \overline{H}^2 \ | \ |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))| \},$$

where $Q = (0,1,0)$, $W = (1, w_1, w_2)$ in $V^*$ (see [4]). It follows that the isometric sphere $I_f$ is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I_f = \left\{ z = (k, t, w')_H \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C} \ | \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

2. We shall give a modified version of the stable basin theorem in [8]. Let

$$B_r = \{ z \in \partial H^2 \ | \ \delta(z,0) < r \},$$

and let $\overline{B}_s = \partial H^2 - \overline{B}_s$. Given $r$ and $s$ with $r < s$, the pair of open sets $(B_r, \overline{B}_s)$ is said to be stable with respect to a set $S$ of elements in $PU(1,2; \mathbb{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_s.$$

A loxodromic element $f$ has a unique complex dilation factor $\lambda(f)$ such that $|\lambda(f)| > 1$. Let $S(r, \epsilon)$ denote the family of loxodromic elements $f$ with fixed points in $B_r$ and $\overline{B}_{1/r'}$, and satisfying $|\lambda(f)-1| < \epsilon$. For positive real numbers $r$ and $r'$ with $r < 1/\sqrt{3}$ and $r' < 1$, we define $\epsilon(r, r')$ by

$$\epsilon(r, r') = \sup\{|\lambda(f)-1|\}, \quad (2.1)$$

where $|\lambda(f)-1|$ satisfies the inequality

$$|\lambda(f)-1| < \sqrt{1 + \left( \frac{1-3\epsilon}{1-2r^2} \right)^2 \left( \frac{1-3r^2}{1-r^2} \right)^2 \left( \frac{r'}{r} \right)^2 - 1}. \quad (2.2)$$

A triple of non-negative numbers $(r, r', \epsilon)$ is said to be a basin point provided that $r < 1/\sqrt{3}$, $r' < 1$ and $\epsilon < \epsilon(r, r')$. In particular, if $r' \leq r$, we call $(r, r', \epsilon)$ a stable basin point. Call the set of all such points the stable basin region. For simplicity, we abbreviate $(r, r', \epsilon)$ to $(r, \epsilon)$. Figure 1 shows our new stable basin region, which contains regions in [1] and [8]. Some stable basin points are tabulated in Table 2.

Exactly the same arguments except for using the following Lemma 2.1 instead of Proposition 3.3 in [1] shows our new stable basin theorem.

Lemma 2.1. Let $b, c > 0$ be given. If $f$ is a complex dilation and its complex dilation factor satisfies $|\lambda(f)-1| \leq \sqrt{1 + (b/c)^2} - 1$, then $f(p) \in B_b(p)$ for $p \in \overline{B}_c$. 
Theorem 2.2 (cf. [8; Stable Basin Theorem]). Given positive real numbers \( r \) and \( r' \) with \( r < 1/\sqrt{3} \) and \( r' < 1 \), the pair of open sets \((B_r, B_{1/r'})\) is stable with respect to the family \( S(r, \epsilon(r, r')) \), where \( \epsilon(r, r') \) is given by (2.2).

Remark 2.3. By arguing as in Corollary 6.14 in [1], we may find the boundary of the stable basin region by equating both sides of inequality (2.2) and solving for \( |\lambda(f) - 1| \) in terms of \( r \). If we use Basmajian and Miner's inequality (6.2) in [1], this involves solving a polynomial of degree 6. Using our inequality (2.2), we have

\[
\frac{a_1 a_2 \sqrt{a_3^2 b_1 + 2 a_3^2 r^2 b_1} - a_2^2 a_1^2 - a_3^2 b_1 r^2}{a_2^2 a_1^2 - a_3^2 b_1 r^4},
\]

where \( a_j = 1 - j r^2 \) and for \( j = 1, 2, 3 \) and \( b_1 = (r'/r)^2 \).

3. We begin with recalling Parker's theorem on the discreteness of subgroups of \( PU(1, 2; \mathbb{C}) \).

Theorem 3.1 ([9; Theorem 2.1]). Let \( g \) be a Heisenberg translation with the form

\[
g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \bar{a} \\ a & 0 & 1 \end{pmatrix},
\]

where \( \text{Re}(s) = \frac{1}{2}|a|^2 \). Let \( f \) be any element of \( PU(1, 2; \mathbb{C}) \) with isometric sphere of radius \( R_f \). If

\[
R_f^2 > \delta(g f^{-1}(\infty), f^{-1}(\infty)) \delta(g f(\infty), f(\infty)) + 2|a|^2,
\]

then the group \( \langle f, g \rangle \) generated by \( f \) and \( g \) is not discrete.

In Theorem 4.5 of [8] we have shown that if \( r < 0.484 \), then Theorem 3.1 leads to the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. By using a more precise estimate on the Heisenberg distance between fixed points of \( f \) in terms of \( R_f \) and \( \lambda(f) \), we have the following same result without the assumption on \( r \).

Theorem 3.2. Fix a stable basin point \((r, \epsilon)\). Let \( g \) be the same element as in Theorem 3.1. Let \( f \) be a loxodromic element with fixed point 0 and \( g \), and satisfying \( |\lambda(f) - 1| < \epsilon \). If \( \delta(0, q) > \frac{\delta(0, g(0))}{R_f}(1 + r^2 + \sqrt{1 + r^2}) \), then the group \( \langle f, g \rangle \) generated by \( f \) and \( g \) is not discrete.

To prove our theorem, we need the following lemmas.

Lemma 3.3. Let \( f \) be a loxodromic element with fixed points 0 and \( g \), satisfying \( |\lambda(f) - 1| < \epsilon \). Then

\[
\left( \frac{\delta(0, q)}{R_f} \right)^2 \leq \frac{2\epsilon - \epsilon^2}{1 - \epsilon}.
\]
Lemma 3.4. For a stable basin point \((r, \epsilon)\),

\[
\frac{1 + r^2 + \sqrt{1 + r^2}}{r^2} > \left(\frac{2\epsilon - \epsilon^2}{1 - \epsilon}\right)^{\frac{1}{2}} \left(2 + \left(8 + \frac{M(\epsilon)}{2}\right)^{\frac{1}{3}}\right),
\]

where \(M(\epsilon) = (1 + \epsilon)^{\frac{1}{2}} + (1 + \epsilon)^{-\frac{1}{2}}\).

![Graph of \(\epsilon(r,r)\)](image)

Figure 1. Graph of \(\epsilon(r,r)\)

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Table 2
References

8. S. Kamiya, On discrete subgroups of $PU(1, 2; C)$ with Heisenberg translations, J. London Math. Soc. (2) 62, 827-842 (2000).

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