<table>
<thead>
<tr>
<th>Title</th>
<th>Local diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke (Studies on complex dynamics and related topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Sumi, Naoya</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2001, 1220: 54-62</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41284">http://hdl.handle.net/2433/41284</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Text version</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Local diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke

Naoya Sumi (鶴見 直哉)

Abstract

We show that if \( f \) is a \( C^2 \)-local diffeomorphism with positive entropy on a \( n \)-dimensional closed manifold \( (n \geq 2) \) then \( f \) is chaotic in the sense of Li-Yorke.

1 Introduction

We study chaotic properties of dynamical systems with positive entropy. Notions of chaos have been given by Li and Yorke [15], Devaney [5] and others. It is well known that if a continuous map of an interval has positive entropy, then the map is chaotic according to the definition of Li and Yorke (cf. [2]). For invertible maps the following holds: let \( f \) be a \( C^2 \)-diffeomorphism of a closed \( C^\infty \)-manifold. If the topological entropy of \( f \) is positive, then \( f \) is chaotic in the sense of Li-Yorke [31].

In this paper we show the following:

Theorem A. Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold. If the topological entropy of \( f \) is positive, then \( f \) is chaotic in the sense of Li-Yorke.

From this theorem we obtain the following corollary.

Corollary B. Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold. If \( f \) is not invertible, then \( f \) is chaotic in the sense of Li-Yorke.

First we shall explain here the definitions and notations used above. Let \( X \) be a compact metric space with metric \( d \) and let \( f : X \rightarrow X \) be a continuous map. A subset \( S \) of \( X \) is a scrambled set of \( f \) if there is a positive number \( \tau > 0 \) such that for any \( x, y \in S \) with \( x \neq y \),

1. \( \lim \sup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > \tau \),
2. \( \lim \inf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0 \).

If there is an uncountable scrambled set \( S \) of \( f \), then we say that \( f \) is chaotic in the sense of Li-Yorke. Li and Yorke showed in [15] that if \( f : [0, 1] \rightarrow [0, 1] \) is a continuous map with a periodic point of period 3, then \( f \) is chaotic in this sense. Note that any scrambled set contains at most one point \( x \) which does not satisfy the following: for any periodic point \( p \in X \),

\[
\lim_{n \rightarrow \infty} \sup d(f^n(x), f^n(p)) > 0.
\]

For another sufficient condition for the chaos in the sense of Li-Yorke, the readers may refer to [4], [7], [8], [9], [10], [11], [19], [20], [34].

Concerning the chaos in the sense of Li-Yorke, Kato introduced the notion of "\( \ast \)-chaos" as follows: let \( F \) be a closed subset of \( X \). A map \( f : X \rightarrow X \) is \( \ast \)-chaotic on \( F \) (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is \( \tau > 0 \) with the property that for any nonempty open subsets \( U \) and \( V \) of \( F \) with \( U \cap V = \emptyset \) and for any natural number \( N \), there is \( n \geq N \) such that \( d(f^n(x), f^n(y)) > \tau \) for some \( x \in U, y \in V \), and
2. for any nonempty open subsets $U, V$ of $F$ and any $\varepsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for some $x \in U, y \in V$.

Such a set $F$ is called a $\ast$-chaotic set. If $S$ is a scrambled set, then the closure of $S$, $\overline{S}$, is a $\ast$-chaotic set. In [10] Kato showed that the converse is true. This is stated precisely as follows:

**Lemma 1** ([10], Theorem 2.4) Let $X$ be a compact metric space and let $F$ be a closed subset of $X$. If $f : X \to X$ is continuous and is $\ast$-chaotic on $F$, then there is an $F_\alpha$-set $S \subset F$ such that $S$ is a scrambled set of $f$ and $\overline{S} = F$. If $F$ is perfect (i.e. $F$ has no isolated points), we can choose $S$ as a countable union of Cantor sets.

By this lemma, to show the existence of uncountable scrambled sets it suffices to show the existence of perfect $\ast$-chaotic sets.

To obtain Theorem A we consider the inverse limit system of $f$. Let $M$ be a closed $C^\infty$-manifold and let $d$ be the distance for $M$ induced by a Riemannian metric $\| \cdot \|$ on $TM$. Let $M^Z$ denote the product topological space $M^Z = \{(x_i) : x_i \in M, i \in \mathbb{Z}\}$. Then $M^Z$ is compact. We define a compatible metric $\tilde{d}$ for $M^Z$ by

$$\tilde{d}((x_i), (y_i)) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}} \quad ((x_i), (y_i) \in M^Z).$$

For $f : M \to M$ a continuous surjection, we let

$$M_f = \{(x_i) : x_i \in M \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\}.$$ 

Then $M_f$ is a closed subset of $M^Z$. The space $M_f$ is called the inverse limit space constructed by $f$. A homeomorphism $\tilde{f} : M_f \to M_f$, which is defined by

$$\tilde{f}((x_i)) = (f(x_i)) \text{ for all } (x_i) \in M_f,$$

is called the shift map determined by $f$. We denote $P^0 : M_f \to M$ the projection defined by $(x_i) \mapsto x_0$. Then $P^0 \circ f = f \circ P^0$ holds. Remark that $f$ is chaotic in the sense of Li-Yorke if and only if so is $\tilde{f}$.

We can show that the topological entropy, $h(f)$, of $f$ coincides with that of $\tilde{f}$. Indeed, for an $f$-invariant probability measure $\nu$, we can find an $\tilde{f}$-invariant probability measure $\mu$ such that $\nu(A) = \tilde{P}^0 \mu(A) = \mu((P^0)^{-1}A))$ for any Borel set $A \subset M$ ([18] Lemma IV 8.3). Let us denote by $h_n(f)$ and $h_\mu(f)$ the metric entropy of $(M, f, \nu)$ and $(M_f, \tilde{f}, \mu)$ respectively. Then we have $h_n(f) = h_{P^0 \mu}(f) = h_\mu(\tilde{f})$ ([25] Lemma 5.2). Therefore, the conclusion is obtained by the variational principle ([32] Theorem 8.6).

We say that a differentiable map $f : M \to M$ is a local diffeomorphism if for $x \in M$ there is an open neighborhood $U_x$ of $x$ in $M$ such that $f(U_x)$ is open in $M$ and $f|_{U_x} : U_x \to f(U_x)$ is a diffeomorphism. Since $M$ is connected, then the cardinal number of $f^{-1}(x)$ is constant. This constant is called the covering degree of $f$. If the covering degree of $f$ is greater than one, $(M_f, M, C, P^0)$ is a fiber bundle where $C$ denotes the Cantor set (see [1] Theorem 6.5.1).

Let $\mu$ be a Borel probability measure on $M_f$ and let $B$ be the Borel $\sigma$-algebra on $M_f$ completed with respect to $\mu$. For $\xi$ a measurable partition of $M_f$ and $\tilde{x} \in M_f$ we denote as $\xi(\tilde{x})$ the element of the partition $\xi$ which contains the point $\tilde{x}$. Then there exists a family $\{\mu^\xi_{\tilde{x}} | \tilde{x} \in M_f\}$ of Borel probability measures satisfying the following conditions:

1. for $\tilde{x}, \tilde{y} \in M_f$ if $\xi(\tilde{x}) = \xi(\tilde{y})$ then $\mu^\xi_{\tilde{x}} = \mu^\xi_{\tilde{y}}$,

2. $\mu^\xi_{\tilde{x}}(\xi(\tilde{x})) = 1$ for $\mu$-almost all $\tilde{x} \in M_f$,

3. for $A \in B$ a function $\tilde{x} \mapsto \mu^\xi_{\tilde{x}}(A)$ is measurable and $\mu(A) = \int_{M_f} \mu^\xi_{\tilde{x}}(A)d\mu(\tilde{x})$.

The family $\{\mu^\xi_{\tilde{x}} | \tilde{x} \in M_f\}$ is called a canonical system of conditional measures for $\mu$ and $\xi$ (see [26] for more details).

To prove Theorem A it suffices to show the following theorem.
Theorem C Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and let \( \mu \) be an \( \tilde{f} \)-invariant ergodic Borel probability measure on \( M_f \).

If the metric entropy of \( \mu \) is positive, then there exists a measurable partition \( \eta \) of \( M_f \) such that \( \supp(\mu^\eta) \) is a perfect \(*\)-chaotic set for \( \mu \)-almost all \( \tilde{\xi} \in M_f \).

Here the support \( \supp(\nu) \) of a finite measure \( \nu \) is the smallest closed set \( C \) with \( \nu(C) = \nu(M_f) \). Equivalently, \( \supp(\nu) \) is the set of all \( \tilde{\xi} \in M_f \) with the property that \( \nu(U) > 0 \) for any open \( U \) containing \( \tilde{\xi} \).

Let us see how Theorem A follows from Theorem C. We know that \( h(\tilde{f}) = \sup\{h_\mu(\tilde{f}) : \mu \in M_e(\tilde{f})\} \) where \( M_e(\tilde{f}) \) is the set of all \( \tilde{f} \)-invariant ergodic probability measures (cf.[27]). Thus, if \( h(\tilde{f}) = h(f) > 0 \), then we can choose \( \mu \in M_e(\tilde{f}) \) with \( h_\mu(\tilde{f}) > 0 \). Therefore, by Theorem C and Lemma 1, \( f \) is chaotic in the sense of Li-Yorke.

2 Key Lemmas

In this section we prepare some lemmas which need to prove Theorem C. Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and \( \mu \) be an \( \tilde{f} \)-invariant ergodic Borel probability measure on \( M_f \) with \( h_\mu(\tilde{f}) > 0 \). As in the previous section we denote as \( B \) the Borel \( \sigma \)-algebra on \( M_f \) completed with respect to \( \mu \). For \( \mu \)-almost all \( \tilde{\xi} = (x_i) \in M_f \), there exist a splitting of the tangent space \( T_{x_0}M = \oplus_{i=1}^{\epsilon(x_0)} E_i(\tilde{\xi}) \) and real numbers \( \lambda_i(x_0) < \lambda_2(x_0) < \cdots < \lambda_s(x_0) \) such that

(a) the maps \( \tilde{\xi} \mapsto E_1(\tilde{\xi}), \lambda_1(x_0) \) and \( s(x_0) \) are measurable, moreover \( E_i(\tilde{f}(\tilde{\xi})) = D_{x_0}f(E_i(\tilde{\xi})) \) and \( \lambda_i(x_0), s(x_0) \) are \( f \)-invariant (\( i = 1, \cdots, s(x_0) \)),

(b) \( \lim_{n \to \infty} \frac{1}{n} \log \|D_{x_0}f^n(\tilde{\xi})\| = \lambda_i(x_0) \) (\( 0 \neq v \in E_i(\tilde{\xi}), i = 1, \cdots, s(\tilde{\xi}) \)) and

(c) \( \lim_{n \to \infty} \frac{1}{n} \log |\det(D_{x_0}f^n(\tilde{\xi}))| = \sum_{i=1}^{s(x_0)} \lambda_i(x_0) \dim E_i(\tilde{\xi}) \)

([21], [33], [29], [30]). The numbers \( \lambda_1(x_0), \cdots, \lambda_s(x_0) \) are called Lyapunov exponents of \( f \) at \( x_0 \). Since \( \mu \) is ergodic, we can put \( s = s(x_0) \), \( \lambda_i = \lambda_i(x_0) \) and \( m_i = \dim E_i(\tilde{\xi}) \) (\( i = 1, \cdots, s \)) for \( \mu \)-almost all \( \tilde{\xi} = (x_i) \in M_f \).

A well-known theorem of Margulis and Ruelle [28] says that entropy is always bounded above by the sum of positive Lyapunov exponents; i.e. \( h_{P^0\mu}(f) \leq \sum_{\lambda_i > 0} \lambda_i m_i \). Since \( \tilde{f} \) has positive entropy, we have \( 0 < h_\mu(\tilde{f}) = h_{P^0\mu}(f) \leq \max(\lambda_i) = \lambda_s \). Fix \( 0 < \lambda < \min(\lambda_i : \lambda_i > 0) \). From [24], [29] and [30] there are measurable functions \( \tilde{\beta} > \tilde{\alpha} > 0 \) and \( \tilde{\gamma} > 1 \) with the following properties: For \( \tilde{\xi} = (x_i) \in M_f \) we put

\[
\tilde{W}_{loc}^u(\tilde{\xi}) = \{ \tilde{y} = (y_i) \in M_f : d(x_0, y_0) \leq \tilde{\alpha}(\tilde{\xi}), d(x_i, y_i) \leq \tilde{\beta}(\tilde{\xi}) e^{-i\lambda} (i \geq 1) \}.
\]

Then

(a) the map \( P^0 \) restricted to \( \tilde{W}_{loc}^u(\tilde{\xi}) \) is injective,

(b) \( P^0(\tilde{W}_{loc}^u(\tilde{\xi})) \) is a \( C^2 \)-submanifold of the ball \( \{ y \in M : d(x_0, y) \leq \tilde{\alpha}(\tilde{\xi}) \} \),

(c) \( T_{x_0}P^0(\tilde{W}_{loc}^u(\tilde{\xi})) = \oplus_{\lambda_i > 0} E_i(\tilde{\xi})(\neq \{0\}) \) for \( \mu \)-almost all \( \tilde{\xi} \in M_f \),

(d) \( d(y_i, z_i) \leq \tilde{\gamma}(\tilde{\xi}) d(y_0, z_0) e^{-i\lambda} \) for \( (y_i), (z_i) \in \tilde{W}_{loc}^u(\tilde{\xi}) \).

For the case when \( f \) is invertible we may refer to [6], [22] and [23].

Let \( \xi \) and \( \eta \) be measurable partitions of \( M_f \). Put \( f^n \xi = \{ f^n C : C \in \xi \} \) for \( n \in \mathbb{Z} \) and then \( (f^n \xi)(\tilde{\xi}) = f^n(\xi(f^{-n}(\tilde{\xi}))) \) for \( \tilde{\xi} \in M_f \). \( \eta \leq \xi \) means that for \( \mu \)-almost all \( \tilde{\xi} \in M_f \) one has \( \xi(\tilde{\xi}) \subset \eta(\tilde{\xi}) \).

Lemma 2 Let \( f \) and \( \mu \) be as above. Then there exists a measurable partition \( \xi \) of \( M_f \) such
(a) $\xi \leq \tilde{f}^{-1}\xi$,

(b) for $\mu$-almost all $\tilde{z} \in M_f$, $\xi(\tilde{z}) \subset \tilde{W}^u_{loc}(\tilde{z})$ and $\xi(\tilde{z})$ contains a neighborhood of $\tilde{z}$ open in $\tilde{W}^u_{loc}(\tilde{z})$,

(c) $\sqrt[n=0]{} \tilde{f}^{-n}\xi$ is the partition into points.

This lemma is similar to [13] Proposition 3.1, [16] Proposition 5.2 and [17] Lemma 2.2. So we omit the proof.

Let $\mathcal{C}$ denote the family of all nonempty closed subsets of $M_f$ and define a metric $d_H$ by

$$d_H(A, B) = \max\{\sup_{b \in B} d(A, b), \sup_{a \in A} d(a, B)\} \quad (A, B \subset C)$$

where $d(A, b) = \inf\{d(a, b) : a \in A\}$. Then it is known that $(\mathcal{C}, d_H)$ is a compact metric space (cf.[12]). If $\xi$ is a measurable partition, then $\tilde{z} \mapsto \xi(\tilde{z}) \in \mathcal{C}$ is measurable. Indeed, this follows from [3] Theorems III.2, III.9, III.22 and the fact that $\{(\tilde{x}, \xi(\tilde{x})) : \tilde{x} \in M_f\}$ is a Borel subset of $M_f \times M_f$. For $A \subset M_f$ we put $\text{diam}(A) = \sup\{d(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in A\}$. Then we have $\text{diam}(A) = \text{diam}(\tilde{A})$. Since $\tilde{z} \mapsto \xi(\tilde{z}) \in \mathcal{C}$ is measurable, $\tilde{z} \mapsto \text{diam}(\xi(\tilde{z}))$ is also a measurable function. By Lemma 2 (c) we have that for $\mu$-almost all $\tilde{z} \in M_f$

$$\text{diam}(\tilde{f}^{-n}\xi(\tilde{z})) \to 0 \quad (1)$$

as $n \to \infty$.

Let $\xi$ and $\eta$ be measurable partitions of $M_f$ and let $\{\mu^\xi_{\tilde{z}} | \tilde{z} \in M_f\}$ be a canonical system of conditional measures for $\mu$ and $\xi$. The mean conditional entropy of $\eta$ with respect to $\xi$ is defined by

$$H_{\mu}(\eta|\xi) = \int - \log \mu^\xi_{\tilde{z}}(\eta(\tilde{x})) d\mu(\tilde{x})$$

(see [27] for details).

**Lemma 3** Let $f$ and $\mu$ be as above and let $\xi$ be as in Lemma 2. Then,

$$h_{\mu}(\tilde{f}) = H_{\mu}(\tilde{f}^{-1}\xi|\xi).$$

For the case when $f$ is invertible this lemma is proved by Ledrappier and Young [14]. We recall that if the covering degree of $f$ is greater than one, then $(M_f, M, C, \nu^\nu)$ is a fiber bundle where $C$ denotes the Cantor set. In view of this fact, the above lemma can be proved by almost the same arguments as the proof of [14] Corollary 5.3 and [16] Corollary 7.1 with some slight modifications. Here we omit the proof.

By Lemma 2(a) we have that $\xi \geq \tilde{f}\xi \geq \tilde{f}^2\xi \geq \cdots$. Let us introduce a measurable partition defined by $\eta = \bigwedge_{n=0}^\infty \tilde{f}^n \xi$. Then we have $\tilde{f}\eta = \eta$. For simplicity put

$$\mu_{\tilde{z}} = \mu_{\tilde{z}}^\eta \quad \text{and} \quad \mu_{\tilde{z}}^\xi = \mu_{\tilde{z}}^\xi(\eta|\xi) \quad (n \in \mathbb{Z}).$$

By Doob's theorem it follows that for a $\mu$-integrable function $\psi : M_f \to \mathbb{R}$

$$\int \psi d\mu_{\tilde{z}} = \lim_{n \to \infty} \int \psi d\mu_{\tilde{z}}^\xi \quad (2)$$

for $\mu$-almost all $\tilde{z}$. Since $\tilde{f}\eta = \eta$ and $\tilde{f}_*\mu = \mu$, by the uniqueness of a canonical system of conditional measures (cf.[26]) we have that for $\mu$-almost all $\tilde{z}$

$$\tilde{f}_*\mu_{\tilde{z}} = \mu_{\tilde{f}\tilde{z}} \quad \text{and} \quad \tilde{f}_*\mu_{\tilde{z}}^\xi = \mu_{\tilde{f}\tilde{z}}^\xi(n+1) \quad (n \in \mathbb{Z}).$$

(3)

Here $(\tilde{f}_* \nu)(A) = \nu(\tilde{f}^{-1}A)$ for a Borel probability measure $\nu$ on $M_f$ and $A \in \mathcal{B}$.

Let $C(M_f)$ be the Banach space of continuous real-valued functions of $M_f$ with the sup norm $| \cdot |_\infty$, and let $\mathcal{M}(M_f)$ be a set of all Borel probability measures on $M_f$ with the weak
topology. Since $C(M_f)$ is separable, there exists a countable set $\{\varphi_1, \varphi_2, \cdots\}$ which is dense in $C(M_f)$. For $\nu, \nu' \in \mathcal{M}(M_f)$ define
\[
\rho(\nu, \nu') = \sum_{n=1}^{\infty} \frac{|\int \varphi_n d\nu - \int \varphi_n d\nu'|}{2^n |\varphi_n|_\infty}.
\]
Then $\rho$ is a compatible metric for $\mathcal{M}(M_f)$ and $(\mathcal{M}(M_f), \rho)$ is compact (cf.[18]). Since (2) holds for $\{\varphi_i\}$, we have
\[
\mu_{\tilde{x}} = \lim_{n \to \infty} \mu_{\tilde{x}}^n
\]
(4) for $\mu$-almost all $\tilde{x}$. For $\nu \in \mathcal{M}(M_f)$ and a measurable partition $\xi$, by the definition of conditional measures $\{\nu_{\tilde{x}}^\xi\}$, the map
\[
M_f \ni \tilde{x} \mapsto \int \varphi_n d\nu_{\tilde{x}}^\xi
\]
is measurable for $n \geq 1$ and thus $\tilde{x} \mapsto \nu_{\tilde{x}}^\xi \in \mathcal{M}(M_f)$ is measurable.

Lemma 4 Let $f, \mu$ and $\{\mu_{\tilde{x}}| \tilde{x} \in M_f\}$ be as above. Then for $\epsilon > 0$ there exists a closed set $F_\epsilon$ with $\mu(F_\epsilon) \geq 1 - \epsilon$ satisfying the map
\[
F_\epsilon \ni \tilde{x} \mapsto \mu_{\tilde{x}} \in \mathcal{M}(M_f)
\]
is continuous.

Proof. Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above and let $\epsilon > 0$. Since $\tilde{x} \mapsto \int \varphi_n d\nu_{\tilde{x}}^\xi$ is measurable for $i \geq 1$, by Lusin's theorem there exists a closed set $F_i$ ($i \geq 1$) with $\mu(F_i) \geq 1 - \epsilon/2^i$ satisfying
\[
F_i \ni \tilde{x} \mapsto \int \varphi_n d\mu_{\tilde{x}} : \text{continuous.}
\]
Then $F_\epsilon = \bigcap_{i=1}^{\infty} F_i$ has the desired property.

For $\nu \in \mathcal{M}(M_f)$ and $E \in B$ let $\nu|_E$ denote the restriction of $\nu$ to $E$, i.e. $\nu|_E(A) = \nu(A \cap E)$ for $A \in B$. Clearly $\nu|_E$ is a finite measure. We denote as $B(\tilde{x}, r)$ and $U(\tilde{x}, r)$ the closed and open balls in $M_f$ with center $\tilde{x} \in M_f$ and radius $r > 0$ respectively. Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above and let $\nu \in \mathcal{M}(M_f)$. For $\tilde{x} \in \text{supp}(\nu)$ and $\epsilon > 0$ we can find $i$ such that
\[
\int_{U(\tilde{x}, \epsilon)} \varphi_i d\nu > \int \varphi_i d\nu - \epsilon.
\]
Since the inequality holds for $\nu'$ sufficiently close to $\nu$, we can easily prove that
\[
\mathcal{M}(M_f) \ni \nu \mapsto \text{supp}(\nu) \in \mathcal{C}
\]
is lower semi-continuous and so the map is measurable ([3] Corollary III.3). Since $\nu \mapsto \text{diam}($supp$(\nu))$ is lower semi-continuous,
\[
\mathcal{P}(M_f) = \{\nu \in \mathcal{M}(M_f) : \nu \text{ is a point measure}\}
\]
\[
\mathcal{P}(M_f) = \{\nu \in \mathcal{M}(M_f) : \text{diam}($supp$(\nu)) = 0\}
\]
is a closed set of $\mathcal{M}(M_f)$. Since $(\tilde{F}^n \xi)(\tilde{x}) \subset \eta(\tilde{x})$, we have
\[
\text{supp}(\mu_{\tilde{x}}^n) \subset \text{supp}(\mu_{\tilde{x}}) \quad (n \in \mathbb{Z})
\]
for $\mu$-almost all $\tilde{x} \in M_f$.

Lemma 5 Let $f, \mu$ and $\{\mu_{\tilde{x}}| \tilde{x} \in M_f\}$ be as above. Then for $\mu$-almost all $\tilde{x} \in M$, supp($\mu_{\tilde{x}}$) has no isolated points.
Theorem

If a point for all large $m' = m'(r)$ is a measurable point, we can take $P = \cap_{j \geq 1} \bigcup_{n \geq j} \tilde{f}^{n}P_{-k}$ and then $\mu(P) = 1$ because $\mu$ is ergodic.

By (3) we have

$$\tilde{f}^{n}(P_{-k}) = \{\tilde{f}^{n}(\tilde{x}) \in M_{f} : \mu^{\tilde{x}}_{-k} \in \mathcal{P}(M)\} = \{\tilde{x} \in M_{f} : \tilde{x}_{-k}^{n} \in \mathcal{P}(M)\} = P_{n-k} \quad (n \in \mathbb{Z}),$$

and so $P = \cap_{j \geq 1} \bigcup_{n \geq j} P_{n-k}$. Thus, for $\tilde{x} \in P$ there exists an increasing sequence $\{n_{i}\}_{i \geq 0}$ such that $\tilde{x} \in P_{n_{i}}$ for $i \geq 0$. Since $\mu_{\tilde{x}} = \lim_{i \to \infty} \mu_{\tilde{x}}^{n_{i}}$ (by (4)) and $\mu^{n_{i}}_{\tilde{x}} \in \mathcal{P}(M_{f})$ for $i$, we have $\mu_{\tilde{x}} \in \mathcal{P}(M_{f})$ for $\tilde{x} \in P$.

Since $\xi \geq \eta$ and $\mu_{\tilde{x}}$ is a point measure for $\mu$-almost all $\tilde{x} \in M_{f}$, so is $\mu_{\tilde{x}}^{\xi}$. Thus $\mu^{\xi}_{\tilde{x}}((\tilde{f}^{-1}\xi)(\tilde{x})) = 1$ for $\mu$-almost all $\tilde{x}$. Therefore

$$h_{\mu}(\tilde{f}) = H_{\mu}(\tilde{f}^{-1}\xi) = \int -\log \mu^{\xi}_{\tilde{x}}(\tilde{f}^{-1}\xi(\tilde{x}))d\mu(\tilde{x}) = 0$$

by Lemma 3. This is a contradiction.

3 Proof of Theorem C

In this section we will prove Theorem C. Let $f$, $\mu$, $\eta$ and $\{\mu_{\tilde{x}}|\tilde{x} \in M_{f}\}$ be as in §2. By Lemma 5, supp($\mu_{\tilde{x}}$) is perfect for $\mu$-almost all $\tilde{x} \in M_{f}$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 1 If $\mu_{\tilde{x}}$ is not a point measure for $\mu$-almost all $\tilde{x} \in M_{f}$, then supp($\mu_{\tilde{x}}$) is a $*-$chaotic set for $\mu$-almost all $\tilde{x} \in M_{f}$.

Proof. The proof of this proposition is similar to that of [31] Proposition 2. However, for completeness we give the proof.

Fix $0 < \epsilon < 1$ and let $F_{\epsilon}$ be as in Lemma 4. By assumption we can take and fix $\tilde{x}_{0} \in \text{supp}(\mu|F_{\epsilon})$ such that $\mu_{\tilde{x}_{0}}$ is not a point measure. Choose two distinct points $\tilde{y}_{1}, \tilde{y}_{2} \in \text{supp}(\mu_{\tilde{x}_{0}})$ and put $\tau = d(\tilde{y}_{1}, \tilde{y}_{2})/2(>0)$. For $0 < \tau < \tau/2$ we can take $\delta = \delta(\tau) > 0$ with

$$\mu(\tilde{y}_{i}, \tau) > \delta \quad (i = 1, 2).$$

Since $U(\tilde{y}_{i}, \tau)$ are open, there exists a large integer $m' = m'(r) > 0$ such that if $\rho(\nu, \mu_{\tilde{x}_{0}}) < 1/m' (\nu \in \mathcal{M}(M_{f}))$, then

$$\nu(U(\tilde{y}_{i}, \tau)) > \delta = \delta(\tau) \quad (i = 1, 2).$$

By Lemma 4 we can find $\epsilon' = \epsilon'(r) > 0$ such that for $\tilde{x} \in U(\tilde{x}_{0}, \epsilon') \cap F_{\epsilon}$

$$\rho(\mu_{\tilde{x}}, \mu_{\tilde{x}_{0}}) < 1/2m' = 1/2m'(r).$$

(6)

Remark that

$$d(U(\tilde{y}_{1}, \tau), U(\tilde{y}_{2}, \tau)) = \inf\{d(\tilde{x}, \tilde{y}) : d(\tilde{x}, \tilde{y}_{1}) < \tau, d(\tilde{y}, \tilde{y}_{2}) < \tau\} > \tau.$$
Let $\xi$ be as in Lemma 2 and put
\[ B_m(n) = \left\{ \tilde{x} \in M_\ell \left| \begin{array}{c}
\rho(\mu_{\tilde{x}}^{[k/2]}, \mu_{\tilde{z}}) < 1/m, \\
\text{diam}(\tilde{f}^{-k+[k/2]}\xi(\tilde{f}^{-k}\tilde{x})) < 1/m \quad (k \geq n)
\end{array} \right. \right\} \]
for $n, m \geq 1$. Then $B_m(n) \subset B_m(n+1)$ and $\mu(\bigcup_{n=0}^\infty B_m(n)) = 1$ by (1) and (4), and so there exists an increasing sequence $\{n_m\}$ such that $\mu(B_m(n_m)) \geq 1 - 1/2^{m+1}$ ($m \geq 1$).
Since $\mu(\bigcap_{k=m}^\infty B_k(n_k)) \geq 1 - 1/2^{m}$ for $m \geq 1$, we can find $D_m \in \mathcal{B}$ with $\mu(D_m) \geq 1 - 2^{-m/2}$ satisfying
\[ \mu_{\tilde{z}}(\bigcap_{k=m}^\infty B_k(n_k)) \geq 1 - 2^{-m/2} \quad (\tilde{x} \in D_m). \] (7)
For $0 < r < \tau/2$ we put
\[ K_r = \bigcap_{k=1}^\infty \bigcup_{m=k}^\infty \left( \bigcap_{n=0}^\infty \tilde{f}^{-\ell}(U(\tilde{x}_0, \epsilon'(r)) \cap F_\epsilon \cap D_m) \right). \]

Since $\mu(U(\tilde{x}_0, \epsilon'(r)) \cap F_\epsilon \cap D_m) \geq \mu(U(\tilde{x}_0, \epsilon'(r)) \cap F_\epsilon) - 2^{-m/2} > 0$ for $m$ sufficiently large, we have $\mu(K_r) = 1$ ($0 < r < \tau/2$) by the ergodicity of $\mu$. Therefore, to obtain the conclusion it suffices to show that $\text{supp}(\mu_{\tilde{z}})$ is a $*$-chaotic set for $\tilde{x} \in K = \cap_{n \geq 1} K_{1/n}$.

To do this fix $\tilde{x} \in K_r$ ($r = 1/n, n \geq 1$) and suppose that nonempty open sets $U_1$ and $U_2$ satisfy
\[ U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \text{supp}(\mu_{\tilde{z}}) \neq \emptyset \quad (j = 1, 2). \]
Choose $m_0 > 0$ with
\[ 0 < 2^{-m_0/2} < \min\{\mu_{\tilde{z}}(U_j) : j = 1, 2\} \quad \text{and} \quad m_0 \geq 2m'. \]
Since $\tilde{x} \in K_r$, by the definition of $K_r$, there exist $m_1 > m_0$ and a sequence of positive integers $\{\ell_k\}_k$ with $\ell_k > n_k$ such that
\[ \tilde{f}^{\ell_k}(\tilde{x}) \in U(\tilde{x}_0, \epsilon'(r)) \cap F_\epsilon \cap D_{m_1} \quad (k \geq 1). \] (8)
Thus, by (3) and (7) we have
\[ \mu_{\tilde{z}}(\tilde{f}^{-\ell_k}(B_k(n_k))) \geq \mu_{\tilde{z}}(\tilde{f}^{-\ell_k}(\bigcap_{k=m}^\infty B_k(n_k))) = \mu_{\tilde{f}^\ell_k(z)}(\bigcap_{k=m}^\infty B_k(n_k)) \geq 1 - 2^{-m_1/2} \geq 1 - 2^{-m_0/2} \quad (k \geq m_1), \]
and so $\mu_{\tilde{z}}(U_j \cap \tilde{f}^{-\ell_k}(B_k(n_k))) \geq \mu_{\tilde{z}}^u(U_j) - 2^{-m_0/2} > 0$. Therefore we can choose
\[ \tilde{z}_j = \tilde{z}_j(k) \in U_j \cap \tilde{f}^{-\ell_k}(B_k(n_k)) \cap \eta(\tilde{z}) \]
for $j = 1, 2$ and $k \geq m_1$.
Since $\tilde{f}^{\ell_k}(\tilde{z}_j) \in B_k(n_k) \cap \tilde{f}^{\ell_k}(\eta(\tilde{z})) \subset B_k(\ell_k) \cap \eta(\tilde{f}^{\ell_k}(\tilde{z}))$, we have
\[ \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{z})} \cap \mu_{\tilde{z}}) < 1/k \leq 1/m_0 \leq 1/2m', \]
\[ \text{diam}(\tilde{f}^{-\ell_k+[\ell_k/2]}\xi(\tilde{f}^{-\ell_k}\tilde{z})) < 1/k \]
for $j = 1, 2$ and $k \geq m_1$. By use of (6) and (8)
\[ \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)} \cap \mu_{\tilde{z}_j}) \leq \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)} \cap \mu_{\tilde{z}_j}) + \rho(\mu_{\tilde{f}^{\ell_k}(\tilde{z})} \cap \mu_{\tilde{z}_j}) < 1/2m' + 1/2m' = 1/m', \]
and so $\mu_{\tilde{f}^{\ell_k+[\ell_k/2]}(\tilde{f}^{-\ell_k}U(\tilde{y}_i, r))} = \mu_{\tilde{f}^{\ell_k}(\tilde{z}_j)} \cap \mu_{\tilde{f}^{-\ell_k}U(\tilde{y}_i, r) \neq \emptyset}$ by (5). Thus we have
\[ \mu_{\tilde{f}^{\ell_k+[\ell_k/2]}(\tilde{f}^{-\ell_k}U(\tilde{y}_i, r))} \cap \mu_{\tilde{f}^{-\ell_k}U(\tilde{y}_i, r) \neq \emptyset} \]
for $1 \leq i, j \leq 2$ and $k \geq m_1$. Since $\bar{z}_j \in U_j$, by (9) we may assume

$$\bar{z}_j \in (\bar{f}^{-t_k + \lfloor t_k/2 \rfloor} \xi)(\bar{z}_j) \subset U_j$$

for $k$ large enough. Therefore

$$U_j \cap \bar{f}^{-t_k} U(\bar{y}_i, r) \supset \bar{f}^{-t_k} U(\bar{y}_i, r) \neq \emptyset$$

for $1 \leq i, j \leq 2$ and $k$ large enough.

Now we take $b_{i,j} = b_{i,j}(k) \in U_j \cap \bar{f}^{-t_k} U(\bar{y}_1, r)$ for $1 \leq i, j \leq 2$ and then

$$b_{i,j} \in U_j \quad (1 \leq i, j \leq 2),$$

$$d(\bar{f}^{t_k}(b_{1,1}), \bar{f}^{t_k}(b_{2,2})) > d(U(\bar{y}_1, r), U(\bar{y}_2, r)) > \tau \quad \text{and}$$

$$d(\bar{f}^{t_k}(b_{1,1}), \bar{f}^{t_k}(b_{1,2})) \leq \text{diam}(U(\bar{y}_1, r)) = 2r = 2/n.$$


