<table>
<thead>
<tr>
<th>Title</th>
<th>On sufficient conditions for Caratheodory functions</th>
</tr>
</thead>
<tbody>
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<td>Yang, Dingong; Owa, Shigeyoshi; Ochiai, Kyohei</td>
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On sufficient conditions for Carathéodory functions

Dingong Yang, Shigeyoshi Owa, and Kyohei Ochiai

Abstract


1 Introduction

Let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in the unit disk $\mathbb{E} = \{z | |z| < 1\}$. If $p(z)$ in $\mathcal{P}$ satisfies $\text{Re}(p(z)) > 0$ for $z \in \mathbb{E}$, then we say that $p(z)$ is the Carathéodory function.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{E}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{E}$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{E}$ such that $|w(z)| \leq |z|$ and $f(z) = g(w(z)) (z \in \mathbb{E})$. If $g(z)$ is univalent in $\mathbb{E}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $\mathbb{E}$. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $\alpha$ in $\mathbb{E}$ if it satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{E})$$

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for some $\alpha \ (0 \leq \alpha < 1)$. We denote by $S^*(\alpha) \ (0 \leq \alpha < 1)$ the subclass of $A$ consisting of all starlike functions of order $\alpha$ in $E$. Also we denote by $S^*(0) = S^*$. For $-1 \leq a \leq 1, \ -1 \leq b \leq 1$ and $a \neq b$, a function $f(z)$ in $A$ is said to be in the class $S^*(a, b)$ if satisfies

$$\frac{zf'(z)}{f(z)} < \frac{1 + az}{1 + bz} \quad (z \in E).$$

The class $S^*(a, b)$ can be reduced to several well known classes of starlike functions by selecting special values for $a$ and $b$. In particular,

$$S^*(1 - 2\alpha, -1) = S^*(2\alpha - 1, 1) = S^*(\alpha) \quad (0 \leq \alpha < 1).$$

For Carathéodory functions, Nunokawa et al. [3] have given the following two theorems.

**Theorem A.** If $p(z) \in P$ satisfies

$$\alpha(p(z))^2 + \beta zp'(z) < \frac{2\alpha + \beta}{2} \left(\frac{1 + z}{1 - z}\right)^2 - \frac{\beta}{2},$$

where $\beta > 0$ and $\alpha > -\frac{\beta}{2}$, then $\text{Re}(p(z)) > 0 \quad (z \in E)$

**Theorem B.** Let $p(z) \in P$ and $w(z)$ be analytic in $E$ with $w(0) = \alpha$ and $w(z) \neq ik \quad (k \in \mathbb{R}, z \in E)$. If

$$\alpha(p(z)) + \beta \frac{zp'(z)}{p(z)} < w(z),$$

where $\alpha > 0$, $\beta > 0$ and $k^2 \geq \beta(2\alpha + \beta)$, then $\text{Re}(p(z)) > 0 \quad (z \in E)$.

For the starlikeness of functions in $A$, the following results have been proved.

**Theorem C** ([4]). If $f(z) \in A$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\text{Re} \left\{zf'(z) \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right\} > -\frac{1}{2} \quad (z \in E),$$

then $f(z) \in S^*$.

**Theorem D** ([1]). If $f(z) \in A$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\text{Re} \left\{zf'(z) \left(\frac{zf''(z)}{f'(z)} + 1\right)\right\} > 0 \quad (z \in E),$$

then $f(z) \in S^* \left(\frac{1}{2}\right)$.

**Theorem E** ([5]). If $f(z) \in A$ satisfies $f'(z) \neq 0$ and

$$\frac{f(z)f''(z)}{(f'(z))^2} < 2 - \frac{2}{(1 - z)^2} \quad (z \in E),$$

then $f(z) \in S^*$.

**Theorem F** ([5]). If $f(z) \in A$ satisfies $f'(z) \neq 0$ and

$$\left|\frac{f(z)f''(z)}{(f'(z))^2}\right| < 2 \quad (z \in E),$$
then $f(z) \in S^\ast$.

In this paper we shall generalize or refine the above results.

To derive our results, we need the following lemma due to Miller and Mocanu [2].

**Lemma.** Let $g(z)$ be analytic and univalent in $E$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $D$ containing $g(E)$, with $\phi(w) \neq 0$ when $w \in g(E)$. Set

$$Q(z) = zg'(z)\phi(g(z)), h(z) = \theta(g(z)) + Q(z)$$

and suppose that

(i) $Q(z)$ is univalent and starlike in $E$, and

(ii) $\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \text{Re} \left\{ \frac{\theta'(g(z))}{\phi(g(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0$ ($z \in E$).

If $p(z)$ is analytic in $E$, with $p(0) = g(0), p(E) \subset D$, and

$$\theta(p(z)) +zp'(z)\phi(p(z)) < \theta(g(z)) + zg'(z)\phi(g(z)) = h(z),$$

then $p(z) \prec g(z)$ is the best dominant of the subordination.

### 2 Main results

Our main result is contained in

**Theorem 1.** Let $a, b, \lambda$ and $\mu$ satisfy either

(i) $0 < a = -b \leq 1, \lambda > -\frac{1}{2}, \mu \in \mathbb{C}$ and $\text{Re}(\mu) \geq 0$, or

(ii) $-1 \leq b < a \leq 1, \lambda \geq 0, \mu \in \mathbb{C}$ and $\text{Re}(\mu) \geq -2\lambda \frac{(1-a)}{(1-b)}$.

If $p(z) \in \mathcal{P}$ and

(1) $\lambda(p(z))^2 + \mu p(z) + zp'(z) \prec h(z),$

where

(2) $h(z) = \frac{a(\lambda a + \mu b)z^2 + (2\lambda a + \mu(a + b) + a - b)z + \lambda + \mu}{(1 + bz)^2}$

then $p(z) \prec \frac{1 + az}{1 + bz}$ and $\frac{1 + az}{1 + bz}$ is the best dominant of (1).

**Proof.** Set

(3) $g(z) = \frac{1 + az}{1 + bz}, \theta(w) = \lambda w^2 + \mu w, \phi(w) = 1.$

Then $g(z)$ is analytic and univalent in $E, g(0) = p(0) = 1, \theta(w)$ and $\phi(w)$ are analytic with $\phi(w) \neq 0$ in the $w$-plane. The function

(4) $Q(z) = zg'(z)\phi(g(z)) = \frac{(a - b)z}{(1 + bz)^2}$
is univalent and starlike in $E$ because

$$\text{Re} \left\{ \frac{z Q'(z)}{Q(z)} \right\} = \text{Re} \left\{ \frac{1 - b z}{1 + b z} \right\} > 0 \quad (z \in E).$$

Further, we have

$$\theta(g(z)) + Q(z) = \lambda \left( \frac{1 + az}{1 + bz} \right)^2 + \mu \frac{1 + az}{1 + bz} + \frac{(a - b)z}{(1 + bz)^2}$$

and

$$\frac{zh'(z)}{Q(z)} = 2\lambda \frac{1 + az}{1 + bz} + \mu + \frac{1 - bz}{1 + bz}.$$

Therefore

(i) For $0 < a = -b \leq 1$, $\lambda > \frac{-1}{2}$, and $\text{Re}(\mu) \geq 0$, it follows from (6) that

$$\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = (2\lambda + 1)\text{Re} \left\{ \frac{1 - bz}{1 + bz} \right\} + \text{Re}(\mu) > 0 \quad (z \in E).$$

(ii) For $-1 \leq b < a \leq 1$, $\lambda \geq 0$, and $\text{Re}(\mu) \geq -2\lambda \frac{(1 - a)}{(1 - b)}$, from (6) we get

$$\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 2\lambda \frac{1 - a}{1 - b} + \text{Re}(\mu) \geq 0 \quad (z \in E).$$

Thus the function $h(z)$ in (5) is close-to-convex and univalent in $E$. From (1) to (5), we see that

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(g(z)) + zg'(z)\phi(g(z)) = h(z).$$

Therefore, by applying the lemma, we conclude that $p(z) \prec g(z)$ and $g(z)$ is the best dominant of (1). The proof of the theorem is complete.

\[\square\]

Remark 1. For $a = -b = 1$, $\lambda = \frac{\alpha}{\beta}$, $\beta > 0$, $\alpha > -\frac{\beta}{2}$, and $\mu = 0$, Theorem 1 (i) coincides with Theorem A by Nunokawa et al [3].

Corollary 1. If $f(z) \in A$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} \right) \prec \frac{(a - b)z}{(1 + bz)^2} \quad (z \in E)$$

for some $a$ and $b$ ($-1 \leq b < a \leq 1$), then $f(z) \in S^*(a, b)$. 
Proof. Let \( p(z) = \frac{zf'(z)}{f(z)} \). Then \( p(z) \in \mathcal{P} \) and (7) can be written as

\[
(8) \quad zp'(z) \prec \frac{(a - b)z}{(1 + bz)^2}
\]

Putting \( \lambda = \mu = 0 \) in Theorem 1 (ii) and using (8), the desired result follows at once.

\[ \square \]

Remark 2. Corollary 1 with \( a = 1 - 2\alpha \) (\( 0 \leq \alpha < 1 \)) and \( b = -1 \) implies that if \( f(z) \in \mathcal{A} \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[
\frac{zf''(z)}{f(z)} \left( 1 + \frac{zf'(z)}{f(z)} - \frac{zf'(z)}{f(z)} \right) < 2(1 - \alpha)\frac{z}{(1 - z)^2},
\]

then \( f(z) \in S^*(\alpha) \) and the order \( \alpha \) is sharp for \( f(z) = \frac{z}{(1 - z)^{2(1 - \alpha)}} \). When \( \alpha = 0 \), this result improves Theorem C by Owa and Obradović [4].

Corollary 2. If \( f(z) \in \mathcal{A} \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[
(9) \quad \frac{z^2f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1) \left( \frac{zf'(z)}{f(z)} \right)^2 \prec \frac{\lambda + z}{(1 - \overline{z})^2} \quad (z \in \mathcal{E})
\]

for some \( \lambda \) (\( \lambda \geq 0 \)), then \( f(z) \in S^* \left( \frac{1}{2} \right) \) and the order \( \frac{1}{2} \) is sharp.

Proof. If we let \( p(z) = \frac{zf'(z)}{f(z)} \), then \( p(z) \in \mathcal{P} \) and it follows from (9) that

\[
(10) \quad \lambda(p(z))^2 + zp'(z) = \frac{z^2f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1) \left( \frac{zf'(z)}{f(z)} \right)^2 < \frac{\lambda + z}{(1 - z)^2}.
\]

Taking \( a = 0 \), \( b = -1 \), \( \lambda \geq 0 \) and \( \mu = 0 \) in Theorem 1 (ii) and using (10), we know that

\( f(z) \in S^* \left( \frac{1}{2} \right) \).

For \( f(z) = \frac{z}{(1 - z)} \), we have

\[
\frac{z^2f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1) \left( \frac{zf'(z)}{f(z)} \right)^2 = \frac{\lambda + z}{(1 - z)^2}
\]

and

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \to \frac{1}{2} \quad \text{as} \quad z \to -1.
\]

Hence the corollary is proved.

\[ \square \]
Remark 3. If we put \( h(z) = \frac{\lambda + z}{(1 - z)^2} \) (\( \lambda > 0 \)), then

\[
h(e^{i\theta}) = -\frac{1 + \lambda \cos \theta - i\lambda \sin \theta}{2(1 - \cos \theta)} \quad (0 < \theta < 2\pi)
\]

and hence

\[
h(\mathbb{E}) = \left\{ w = u + iv : v^2 > -\frac{\lambda^2}{1 + \lambda} \left( u - \frac{\lambda - 1}{4} \right) \right\},
\]

which properly contains the half plane \( \text{Re}(w) > \frac{\lambda - 1}{4} \). Thus Corollary 2 with \( \lambda = 1 \) improves Theorem D by Li and Owa [1].

Corollary 3. Let \(-1 \leq b < a \leq 1\) and \(\text{Re}(\mu) \geq 0\). If \(f(z) \in \mathcal{A}\) satisfies \(f'(z) \neq 0\) and

\[
(1 - \mu) \frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} < h(z) \quad (z \in \mathbb{E}),
\]

where

\[
h(z) = \frac{b(b - \mu a)z^2 + (3b - a - \mu(a + b))z + 1 - \mu}{(1 + bz)^2},
\]

then \(f(z) \in S^*(b, a)\).

Proof. Let us define \(p(z)\) in \(\mathbb{E}\) by

\[
p(z) = \frac{f(z)}{zf'(z)}.
\]

Then \(p(z) \in \mathcal{P}\) and it follows from (11), (12) and (13) that

\[
\mu p(z) + z p'(z) = 1 + (\mu - 1) \frac{f(z)}{zf'(z)} - \frac{f(z)f''(z)}{(f'(z))^2} < \frac{\mu abz^2 + (\mu(a + b) + a - b)z + \mu}{(1 + bz)^2} \quad (z \in \mathbb{E}).
\]

Therefore, by applying Theorem 1 (ii) with \(\lambda = 0\) and \(\text{Re}(\mu) \geq 0\), we have

\[
p(z) = \frac{f(z)}{zf'(z)} < \frac{1 + az}{1 + bz}.
\]

This implies that \(f(z) \in S^*(b, a)\). \(\square\)

Remark 4. Letting \(a = 1\), \(b = -1\) and \(\mu = 1\) in Corollary 3, we get Theorem E by Tuneski [5].

For \(a = 1\), \(b = 0\) and \(\mu = 1\), Corollary 3 lead to

Corollary 4. If \(f(z) \in \mathcal{A}\) satisfies \(f'(z) \neq 0\) and

\[
\left| \frac{f(z)f''(z)}{(f'(z))^2} \right| < 2 \quad (z \in \mathbb{E}),
\]
then \( f(z) \in S^* \left( \frac{1}{2} \right) \) and the order \( \frac{1}{2} \) is sharp for the function \( f(z) = \frac{z}{1 - z} \).

**Remark 5.** Corollary 4 refines Theorem F by Tuneski [5].

Taking \( a = 0 \), \( b = -c \) and \( \mu = 1 \) in Corollary 3, we have

**Corollary 5.** If \( f(z) \in \mathcal{A} \) satisfies \( f'(z) \neq 0 \) and

\[
\frac{f(z)f''(z)}{(f'(z))^2} < 1 - \frac{1}{(1 - cz)^2} \quad (z \in \mathcal{E})
\]

for some \( c (0 < c \leq 1) \), then

\[
(14) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < c \quad (z \in \mathcal{E}).
\]

The bound \( c \) in (14) is sharp for the function \( f(z) = ze^{-cz} \).

Next we derive

**Theorem 2.** Let \(-1 \leq b < a \leq 1 \), \( \lambda \geq 0 \) and \( \mu \geq -\frac{1 - a}{1 - b} \). If \( p(z) \in \mathcal{P} \) with \( p(z) \neq -\mu \) \((z \in \mathcal{E})\) and

\[
(15) \quad \lambda p(z) + \frac{zp'(z)}{p(z) + \mu} < h(z) \quad (z \in \mathcal{E}),
\]

where

\[
h(z) = \frac{\lambda acz^2 + (\lambda(a + c) + c - b)z + \lambda}{(1 + bz)(1 + cz)}, \quad c = \frac{a + \mu b}{1 + \mu},
\]

then \( p(z) \prec \frac{1 + az}{1 + bz} \) and \( \frac{1 + az}{1 + bz} \) is the best dominant of (15).

**Proof.** We choose

\[
g(z) = \frac{1 + az}{1 + bz}, \quad \theta(w) = \lambda w, \quad \phi(w) = \frac{1}{w + \mu}
\]

and \( \mathbb{D} = w : w \neq -\mu \) in the Lemma. Noting that

\[
(16) \quad \text{Re} \left( g(z) \right) > \frac{1 - a}{1 - b} \geq -\mu \quad (z \in \mathcal{E}),
\]

the function \( \phi(w) \) is analytic in \( \mathbb{D} \) containing \( g(\mathcal{E}) \). From (16) we see that

\[
1 + \mu > 0, \quad -1 \leq b < c = \frac{a + \mu b}{1 + \mu} \leq 1.
\]

The function

\[
Q(z) = zg'(z)\phi(g(z)) = \frac{(c - b)z}{(1 + bz)(1 + cz)}
\]

is univalent and starlike in \( \mathcal{E} \) because

\[
\text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} = -1 + \text{Re} \left( \frac{1}{1 + bz} \right) + \text{Re} \left( \frac{1}{1 + cz} \right)
\]
for $z \in \mathbb{E}$. Further, we have
\[
\theta(g(z)) + Q(z) = \lambda \left( \frac{1 + az}{1 + bz} + \frac{(c - b)z}{(1 + bz)(1 + cz)} \right) = h(z)
\]
and
\[
\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \lambda (1 + \mu) \text{Re} \left( \frac{1 + cz}{1 + bz} \right) + \text{Re} \left( \frac{zQ'(z)}{Q(z)} \right)
\]
\[
> \lambda (1 + \mu) \frac{1 - c}{1 - b} \geq 0 \quad (z \in \mathbb{E})
\]
for $\lambda \geq 0$. The other conditions of the lemma are seen to be satisfied. Hence $p(z) < g(z)$ and $g(z)$ is the best dominant of (15). The proof is complete.

\[\square\]

**Remark 6.** Note that the univalent function $h(z)$ defined by
\[
h(z) = \frac{az^2 + 2(\alpha + \beta)z + \alpha}{1 - z^2} = \frac{1 + z}{1 - z} + 2z \frac{z}{1 - z^2} \quad (\alpha > 0, \beta > 0)
\]
maps $\mathbb{E}$ onto the complex plane minus the half-lines
\[
l_1 = w = u + iv : u = 0, \quad v \leq \sqrt{\beta(2\alpha + \beta)}
\]
and
\[
l_2 = w = u + iv : u = 0, \quad v \leq -\sqrt{\beta(2\alpha + \beta)}.
\]
For $a = 1$, $b = -1$, $\lambda = \alpha \beta$, $\alpha > 0$, $\beta > 0$ and $\mu = 0$, Theorem 2 reduces to Theorem B by Nunokawa et al [3].

Theorem 2 with $\mu = 0$ and $p(z) = \frac{zf'(z)}{f(z)}$ leads to the following corollary.

**Corollary 6.** Let $-1 \leq b < a \leq 1$ and $\lambda \geq 0$. If $f(z) \in \mathcal{A}$ satisfies $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ and
\[
(\lambda - 1) \frac{zf'(z)}{f(z)} + 1 + \frac{zf''(z)}{f'(z)} \prec h(z) \quad (z \in \mathbb{E}),
\]
where
\[
h(z) = \frac{\lambda a^2 z^2 + (2\lambda a + a - b)z + \lambda}{(1 + az)(1 + bz)},
\]
then $f(z) \in \mathcal{S}^*(a, b)$. 
References


*Dinggong Yang
Department of Mathematics
Suzhou University
Suzhou, Jiangsu 215006
People’s Republic of China*

*Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
e-mail: owa@math.kindai.ac.jp*

*Kyohei Ochiai
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
e-mail: ochiai@math.kindai.ac.jp*