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<th>Title</th>
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</thead>
<tbody>
<tr>
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</tr>
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Upper bound of the best constant of the Trudinger-Moser inequality and its application to the Gagliardo-Nirenberg inequality

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We consider the best constant of the Trudinger-Moser inequality in \( \mathbb{R}^n \). Let \( \Omega \) be an arbitrary domain in \( \mathbb{R}^n \). It is well known that the Sobolev space \( H_0^{1/p,p}(\Omega) \), \( 1 < p < \infty \), is continuously embedded into \( L^q(\Omega) \) for all \( q \) with \( p \leq q < \infty \). However, we cannot take \( q = \infty \) in such an embedding. For bounded domains \( \Omega \), Trudinger [18] treated the case \( p = n(\geq 2) \), i.e., \( H_0^{1,n}(\Omega) \) and proved that there are two constants \( \alpha \) and \( C \) such that

\[
\| \exp(\alpha|u|^n) \|_{L^1(\Omega)} \leq C|\Omega| \quad (0.1)
\]

holds for all \( u \in H_0^{1,n}(\Omega) \) with \( \|\nabla u\|_{L^n(\Omega)} \leq 1 \). Here and hereafter \( p' \) represents the Hölder conjugate exponent of \( p \), i.e., \( p' = p/(p-1) \). Moser [9] gave the optimal constant for \( \alpha \) in (0.1), which shows that one cannot take \( \alpha \) greater than \( 1/(n^{n-2} \omega_n^{n-1}) \), where \( \omega_n \) is the volume of the unit \( n \)-ball, that is, \( \omega_n : = |B_1| = 2\pi^{n/2}/(n\Gamma(n/2)) \) (\( \Gamma \): the gamma function). Adams [2] generalized Moser's result to the case \( H_0^{m,n/m}(\Omega) \) for positive integers \( m < n \) and obtained the sharp constant corresponding to (0.1).

When \( \Omega = \mathbb{R}^n \), Ogawa [10] and Ogawa-Ozawa [11] treated the Hilbert space \( H^{n/2,2}(\mathbb{R}^n) \) and then Ozawa [14] gave the following general embedding theorem in the Sobolev space \( H^{n/p,p}(\mathbb{R}^n) \) of the fractional derivatives which states that

\[
\| \Phi_\nu(\alpha|u|^{p'}) \|_{L^1(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}^{p'/(p'-1)} \quad (0.2)
\]
holds for all \( u \in H^{n/p,p}(\mathbb{R}^n) \) with \( \|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leq 1 \), where

\[
\Phi_p(\xi) = \exp(\xi) - \sum_{j=0}^{j_p-1} \frac{\xi^j}{j!} = \sum_{j=j_p}^{\infty} \frac{\xi^j}{j!}, \quad j_p := \min\{j \in \mathbb{N} \mid j \geq p - 1\}.
\]

The advantage of (0.2) gives the scale invariant form. Concerning the sharp constant for \( \alpha \) in (0.2), Adachi-Tanaka [1] proved a similar result to Moser's in \( H^{1,n}(\mathbb{R}^n) \).

Our purpose is to generalize Adachi-Tanaka's result to the space \( H^{n/p,p}(\mathbb{R}^n) \) of the fractional derivatives. We show an upper bound of the constant \( \alpha \) in (0.2). Indeed, the following theorem holds:

**Theorem 0.1.** Let \( 2 \leq p < \infty \). Then, for every \( \alpha \in (A_p, \infty) \), there exists a sequence \( \{u_k\}_{k=1}^{\infty} \subset H^{n/p,p}(\mathbb{R}^n) \setminus \{0\} \) with \( \|(-\Delta)^{n/(2p)}u_k\|_{L^p(\mathbb{R}^n)} \leq 1 \) such that

\[
\frac{\|\Phi_p(\alpha|u_k|^p')\|_{L^1(\mathbb{R}^n)}}{\|u_k\|^p_{L^p(\mathbb{R}^n)}} \to \infty \quad \text{as} \quad k \to \infty,
\]

where \( A_p \) is defined by

\[
A_p := \frac{1}{\omega_n} \left[ \frac{\pi^{n/2}2^{n/p}\Gamma(n/(2p))}{\Gamma(n/(2p'))} \right]^{p'}.
\] (0.3)

**Remark.** Let \( \alpha_p \) be the best constant of (0.2), i.e.,

\[
\alpha_p := \sup\{\alpha > 0 \mid \text{The inequality (0.2) holds with some constant } C,\}.
\]

Then Theorem 0.1 implies that \( \alpha_p \leq A_p \) for \( 2 \leq p < \infty \).

Next, if we give a similar type estimate to (0.2) by taking another normalization such as \( \|(I-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leq 1 \), then we can cover all \( 1 < p < \infty \). Moreover, when \( p = 2 \), it turns out that our constant \( A_2 \) of (0.3) is optimal. To state our second result, let us recall the rearrangement \( f^* \) of the measurable function \( f \) on \( \mathbb{R}^n \). For detail, see Section 2 (Stein-Weiss [16]). We denote by \( f^{**} \) the average function of \( f^* \), i.e.,

\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau)d\tau \quad \text{for } t > 0.
\]

Our theorem now reads:
**Theorem 0.2.** Let $1 < p < \infty$ and $A_p$ be as in (0.3).

(i) For every $\alpha \in (A_p, \infty)$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset H^{n/p,p}(\mathbb{R}^n)$ with $\|(I - \Delta)^{n/(2p)}u_k\|_{L^p(\mathbb{R}^n)} \leq 1$ such that

$$\|\Phi_p(\alpha |u_k|^p')\|_{L^1(\mathbb{R}^n)} \to \infty \text{ as } k \to \infty.$$ 

(ii) We define $A_p^*$ by

$$A_p^* = A_p/B_p^{1/(p-1)},$$

where

$$B_p := (p-1)^p \sup \left\{ \int_0^\infty (f^{**}(t) - f^*(t))^p dt \mid \|f\|_{L^p(\mathbb{R}^n)} \leq 1 \right\}.$$ 

Then for every $\alpha \in (0, A_p^*)$, there exists a positive constant $C$ depending only on $p$ and $\alpha$ such that

$$\|\Phi_p(\alpha |u|^p')\|_{L^1(\mathbb{R}^n)} \leq C$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(I - \Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leq 1$.

**Remark.** Later, we shall show that

$$1 \leq B_p \leq p^p - (p-1)^p \text{ for } 1 < p < \infty.$$ 

In particular, for $2 \leq p < \infty$, there holds

$$B_p = (p-1)^{p-1}. \quad (0.5)$$

In any case, we obtain $A_p^* \leq A_p$ for $1 < p < \infty$.

Since it follows from (0.5) that $B_2 = 1$, we see that $A_2 = A_2^* = (2\pi)^n/\omega_n$ is the best constant of (0.4). Hence, the following corollary holds:

**Corollary 0.1.** (i) For every $\alpha \in ((2\pi)^n/\omega_n, \infty)$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset H^{n/2,2}(\mathbb{R}^n)$ with $\|(I - \Delta)^{n/4}u_k\|_{L^2(\mathbb{R}^n)} \leq 1$ such that

$$\|\Phi_2(\alpha |u_k|^2)\|_{L^1(\mathbb{R}^n)} \to \infty \text{ as } k \to \infty.$$ 

(ii) For every $\alpha \in (0, (2\pi)^n/\omega_n)$, there exists a positive constant $C$ depending only on $\alpha$ such that

$$\|\Phi_2(\alpha |u|^2)\|_{L^1(\mathbb{R}^n)} \leq C$$

holds for all $u \in H^{n/2,2}(\mathbb{R}^n)$ with $\|(I - \Delta)^{n/4}u\|_{L^2(\mathbb{R}^n)} \leq 1$. 


It seems to be an interesting question whether or not (0.6) does hold for $\alpha = (2\pi)^n/\omega_n$.

Next, we consider the Gagliardo-Nirenberg interpolation inequality which is closely related to the Trudinger-Moser inequality. Ozawa [14] proved that for $1 < p < \infty$ there is a constant $M$ depending only on $p$ such that

$$
\|u\|_{L^q(\mathbb{R}^n)} \leq M q^{1/p'} \|u\|_{L^p(\mathbb{R}^n)}^{p/q} \|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)}^{1-p/q}
$$

(0.7)

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ and for all $q \in [p, \infty)$. Ozawa [13],[14] also showed the fact that (0.2) and (0.7) are equivalent and he gave the relation between $\alpha$ in (0.2) and $M$ in (0.7). Combining his formula with our result, we obtain an estimate of $M$ from below. Indeed, there holds the following theorem:

**Theorem 0.3.** Let $2 \leq p < \infty$. We define $M_p$ and $m_p$ as follows.

$$
M_p := \inf\{M > 0 \mid \text{The inequality (0.7) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [p, \infty)\},
$$

$$
m_p := \inf\{M > 0 \mid \text{There exists } q_0 \in [p, \infty) \text{ such that the inequality (0.7) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [q_0, \infty)\}.
$$

Then there holds

$$
M_p \geq m_p \geq \frac{1}{(p'eA_p^{1/p'})^{1/p'}}.
$$

Since Ozawa [13],[14] gave the relation between the constants $\alpha$ in (0.2) and $M$ in (0.7), we obtain a lower bound of the best constant for the Sobolev inequality in the critical exponent:

**Theorem 0.4.** Let $1 < p < \infty$.

(i) For every $M > (p'eA_p)^{-1/p'}$, there exists $q_0 \in [p, \infty)$ depending only on $p$ and $M$ such that

$$
\|u\|_{L^q(\mathbb{R}^n)} \leq M q^{1/p'} \|(I - \Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)}
$$

(0.8)

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ and for all $q \in [q_0, \infty)$.

(ii) We define $\overline{M}_p$ and $\overline{m}_p$ as follows.

$$
\overline{M}_p := \inf\{M > 0 \mid \text{The inequality (0.8) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [p, \infty)\},
$$

$$
\overline{m}_p := \inf\{M > 0 \mid \text{There exists } q_0 \in [p, \infty) \text{ such that the inequality (0.8) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [q_0, \infty)\}.
$$
Then there holds
\[ \overline{M}_p \geq \overline{m}_p \geq \frac{1}{(p'eA_p)^{1/p'}}. \]

Since we have obtained \( A_2 = A_2^* \) for \( p = 2 \), we see that
\[ \frac{1}{\sqrt{2eA_2}} = \frac{1}{\sqrt{2eA_2^*}} = \sqrt{\frac{\omega_n}{2^{n+1}e\pi^n}}. \]

Hence, the above theorem gives the best constant for (0.8). Indeed, we have the following corollary:

**Corollary 0.2.** (i) For every \( M > \sqrt{\omega_n/(2^{n+1}e\pi^n)} \), there exists \( q_0 \in [2, \infty) \) such that
\[ \|u\|_{L^q(\mathbb{R}^n)} \leq M q_0^{1/2} \|(I - \Delta)^{n/4}u\|_{L^2(\mathbb{R}^n)} \]
holds for all \( u \in H^{n/2,2}(\mathbb{R}^n) \) and for all \( q \in [q_0, \infty) \).

(ii) For every \( 0 < M < \sqrt{\omega_n/(2^{n+1}e\pi^n)} \) and \( q \in [2, \infty) \), there exist \( q_0 \in [q, \infty) \) and \( u_0 \in H^{n/2,2}(\mathbb{R}^n) \) such that
\[ \|u_0\|_{L^{q_0}(\mathbb{R}^n)} > M q_0^{1/2} \|(I - \Delta)^{n/4}u_0\|_{L^2(\mathbb{R}^n)} \]
holds.

To prove our theorems, by means of the Riesz and the Bessel potentials, we first reduce the Trudinger-Moser inequality to some equivalent form of the fractional integral. The technique of symmetric decreasing rearrangement plays an important role for the estimate of fractional integrals in \( \mathbb{R}^n \). To this end, we make use of O’Neill’s result [12] on the rearrangement of the convolution of functions. Such a procedure is similar to Adams [2]. First, we shall show that for every \( \alpha \in (0, A_p^*) \), there exists a positive constant \( C \) depending only on \( p \) and \( \alpha \) such that (0.4) holds for all \( u \in H^{n/p,p}(\mathbb{R}^n) \) with
\[ \|(I - \Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leq 1. \] On the other hand, we shall show that the constant \( \alpha \) holding (0.2) and (0.4) in \( \mathbb{R}^n \) can be also available for the corresponding inequality in bounded domains. Since Adams [2] gave the sharp constant \( \alpha \) in the corresponding inequality to (0.1), we obtain an upper bound \( A_p \) as in (0.3). For general \( p \), we have \( A_p^* \leq A_p \). In particular, for \( p = 2 \), there holds \( A_2^* = A_2 \), which provides us the best constant of (0.4).
References


