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Kyoto University
Small global solutions for the nonlinear Dirac equation

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1 Introduction

In this note we study the Cauchy problem for the nonlinear Dirac equation (NLD) in space-time \( \mathbb{R}^{1+n} \):

\[
\partial_t \psi = (iA_0 + \sum_{j=1}^{n} A_j \partial_j) \psi + \lambda |(A_0 \psi | \psi)|^{(p-1)/2} \psi,
\]

\( \psi(0) = \phi \),

where \( \psi : \mathbb{R}^{1+n} \rightarrow \mathbb{C}^{N(n)} \) is a function of \( (t, x) \in \mathbb{R} \times \mathbb{R}^n \), \( \phi : \mathbb{R}^n \rightarrow \mathbb{C}^{N(n)} \) is a given Cauchy data, \( \lambda \in \mathbb{C} \) and \( p > 1 \) are constants, \( \partial_t = \partial/\partial t \), \( \partial_j = \partial/\partial x_j \) with space variable \( x = (x_1, \ldots, x_n) \), \( (\cdot | \cdot) \) denotes the inner product in \( \mathbb{C}^{N(n)} \). \( A_0, \ldots, A_n \) denote the \( N(n) \times N(n) \) matrices satisfying \( A_i A_j + A_j A_i = 2 \delta_{ij} \) \( I \), where \( \delta_{ij} \) is Kronecker's delta and \( I \) is the unit matrix. \( N(n) \) is an integer depending on the space dimension \( n \).

There are several ways to construct the set of matrices satisfying the anticommutation relation above. The set of \( (A_0^{(n)}, \ldots, A_n^{(n)}) \) for \( n \) dimensional case can be derived from \( (A_0^{(n-1)}, \ldots, A_{n-1}^{(n-1)}) \) for \( n-1 \) dimensional case inductively.

Example 1 For \( n = 1 \), \( A_0^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( A_1^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

For \( n \geq 2 \), \( A_j^{(n)} = \begin{pmatrix} 0 & A_{j-1}^{(n-1)} \\ A_j^{(n-1)} & 0 \end{pmatrix} \), \( j = 0, \ldots, n-1 \), \( A_n^{(n)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \).

Then \( N(n) = 2^n \).

Example 2 For \( n = 1 \), \( A_0^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( A_1^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Let \( m \) be an integer.

For \( n = 4m + 2 \), \( A_j^{(n)} = A_j^{(n-1)} \), \( j = 0, \ldots, n-1 \), \( A_n^{(n)} = i A_0^{(n-1)} \cdots A_{n-1}^{(n-1)} \).

For \( n = 4m + 1, 4m + 3 \), \( A_j^{(n)} = \begin{pmatrix} 0 & A_{j-1}^{(n-1)} \\ A_j^{(n-1)} & 0 \end{pmatrix} \), \( j = 0, \ldots, n-1 \), \( A_n^{(n)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \).

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For $n = 4m + 4$, \( A_j^{(n)} = A_j^{(n-1)} \), \( j = 0, \ldots, n-1 \), \( A_n^{(n)} = A_0^{(n-1)} \cdots A_{n-1}^{(n-1)} \).

Then \( N(n) = 2^{[(n+1)/2]} \), where \([a]\) denotes the largest integer which is less than or equal to \(a\).

We consider the global existence of solution with small data for \( NLD \). The case of \( n = 3 \) have been already studied in [3], [7]. Our basic tool for the proof is Strichartz estimate for Klein–Gordon equation which works for the space-time norm \( L_t^q B_x \), \( q \geq 2 \), where \( B_x \) denotes suitable Besov spaces on \( \mathbb{R}^n \). These estimates have been studied for \( q > 2 \), though the estimates for \( q = 2 \) i.e. \( L_t^2 B_x \) have been excluded until lately and play an important role for our results. First study on the estimate for \( q = 2 \) was given by Lindblad and Sogge [6] and Ginibre and Velo [4] independently in 1995 for the wave equation. Keel and Tao [5] proved the end point estimate in 1998 for wave and Schrödinger equations. For Klein–Gordon equation, estimate on \( q = 2 \) can be found in [7]. In this note we give estimates applicable to more general norm in space variables.

Before stating our results, we shall give a scaling approach in this problem. For instance let us consider the massless case of \( NLD \):

\[
\partial_t \psi = \sum_{j=1}^{n} A_j \partial_j \psi + \lambda |(A_0 \psi | \psi)^{(p-1)/2} \psi.
\]  

(1.2)

We scale the function \( \psi \) in the form

\[
\psi_\gamma(t, x) = \gamma^{-\frac{1}{p-1}} \psi(\gamma t, \gamma x), \quad \gamma > 0.
\]  

(1.3)

Then we see that \( \psi_\gamma \) is a solution of (1.2) if and only if \( \psi \) is a solution of (1.2). We take the initial data belonging to the homogeneous Sobolev space \( \tilde{H}^s \),

\[
\|\psi_\gamma(0)\|_{\tilde{H}^s} = \gamma^{s-n/2+1/(p-1)} \|\psi(0)\|_{\tilde{H}^s}.
\]  

(1.4)

Therefore we may think \( s(p) := n/2 - 1/(p-1) \) as a critical exponent for \( NLD \).

Now we give our results.

**Theorem 3** Let \( n, p, \phi \) satisfy the following conditions:

1. \( n = 1, \quad p \geq 5, \quad \|\phi\|_{B_{2,1}^1} \ll 1, \quad s = 1/2 + 1/(p-1), \)
2. \( n = 2, \quad 3 < p \leq 5, \quad \|\phi\|_{B_{2,1}^1} \ll 1, \quad s = 1/2 + 1/(p-1), \)
3. \( n = 2, \quad p > 5, \quad \|\phi\|_{\tilde{H}^s} \ll 1, \quad s > 1, \)
4. \( n = 3, \quad p = 3, \quad \|\phi\|_{\tilde{H}^s} \ll 1, \quad s > 1, \)
5. \( n = 3, \quad p > 3, \quad \|\phi\|_{\tilde{H}^s} \ll 1, \)
6. \( n \geq 4, \quad p = 3, \quad \|\phi\|_{B_{2,1}^3} \ll 1, \)
7. \( n \geq 4, \quad p > 3, \quad \|\phi\|_{\tilde{H}^s} \ll 1, \quad (s(p) < (p-1)/2 \quad \text{if} \quad p \neq \text{odd}). \)

Then \( NLD \) has a solution \( \psi \in C(\mathbb{R}; X) \), where \( X \) denotes the space of data indicated above.
Remark The cases (4), (5) were proved in [7], [3] respectively.

We close this section by introducing some notation. For any $r$ with $1 \leq r \leq \infty$, $L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any $r$ with $1 < r < \infty$, $H^s_r$ [resp. $\dot{H}^s_r$] denotes the inhomogeneous [resp. homogeneous] Sobolev space. For any $s \in \mathbb{R}$ and any $r, m$ with $1 \leq r, m \leq \infty$, $B^s_{r,m}$ [resp. $B^s_{r,m}$] denotes the inhomogeneous [resp. homogeneous] Besov space. We make abbreviations such as $H^s = H^s_2$, $\dot{H}^s = \dot{H}^s_2$, and $L^q = L^q(\mathbb{R})$.

For any $s \in \mathbb{R}$ and any $r$ with $1 < r < \infty$, $H^s_{r,2}$ [resp. $\dot{H}^s_{r,2}$] denotes the inhomogeneous [resp. homogeneous] Sobolev space. For any $s \in \mathbb{R}$ and any $r, m$ with $1 \leq r, m \leq \infty$, $B^s_{r,m}$ [resp. $B^s_{r,m}$] denotes the inhomogeneous [resp. homogeneous] Besov space. We make abbreviations such as $H^s = H^s_2$, $\dot{H}^s = \dot{H}^s_2$, and $L^q = L^q(\mathbb{R})$.

Occasionally we use $\sim< C$ to mean $C$, where $C$ is a positive constant.

We give some properties for Sobolev and Besov spaces which seem to be important for following argument (see [1]).

$$H^\alpha \hookrightarrow H^\beta, \quad B^\alpha_{r,m} \hookrightarrow B^\beta_{q,m} \quad (1.5)$$

with $\alpha - n/p = \beta - n/q$, $\alpha \geq \beta$.

$$H^\alpha_p \hookrightarrow B^\alpha_{r,2} \quad \text{[resp. } B^\alpha_{p,2} \hookrightarrow H^\alpha_p \text{]} \quad (1.6)$$

with $p \leq 2$ [resp. $p \geq 2$].

$$B^\alpha_{2,1} \hookrightarrow B^\alpha_{2,2} = H^\alpha \hookrightarrow B^\beta_{2,1} \quad (1.7)$$

with $\alpha > \beta$. We often use the embedding

$$B^0_{\infty,1} \hookrightarrow L^\infty. \quad (1.8)$$

2 Proof

We employ a contraction argument to obtain the solution. For this purpose, we prepare two lemmas, Strichartz estimate and interpolation estimate for $L^\infty$ norm.

For simplicity we set $f(\psi) = \lambda |(A_0 \psi |\psi)|^{p-1/2} \psi$, and $\omega = (1 - \Delta)^{1/2}$. The solutions for NLD satisfy the following integral equation (see [7]):

$$\psi(t) = U(t)\phi + \int_0^t U(t-t')f(\psi(t'))dt' \quad (2.1)$$

with $U(t) = \cos t\omega + (iA_0 + \sum A_j \partial_j)\omega^{-1} \sin t\omega$. We investigate the operator $U(t)$. We give the following lemma which is often called Strichartz estimate.

**Lemma 4** Let $k = 1, 2$. The following estimate holds.

$$\|U(t)u\|_{L^q(\mathbb{R};B^{-s}_{r,k})} \lesssim \|u\|_{B^s_{r,k}}, \quad (2.2)$$

where $0 \leq 1/q \leq 1/2$, $0 \leq 1/r \leq 1/2 - 2/(n - 1 + \theta)q$, $(n + \theta, q) \neq (3, 2)$ and

$$\frac{n}{2} - \frac{n - 1 - \theta}{n - 1 + \theta} \frac{1}{q} = \sigma - \frac{n}{r} = 0, \quad (2.3)$$

for $0 \leq \theta \leq 1$. 
From (2.2), we have the homogeneous estimate as follows,
\[
\left\| \int_0^t U(t-t')f(t')dt' \right\|_{L^q(R; B^{-\sigma}_{r,k})} \leq \int_{-\infty}^{\infty} \left\| U(t)U(-t')f(t') \right\|_{L^q(R; B^{-\sigma}_{r,k})} dt',
\]
(2.4)

**Remark** (i) The estimate for \( k = 2 \) was proved in [7]. (ii) We may replace \( K(t) = e^{\pm it\omega} \) for \( U(t) \). (iii) From the condition (2.3), if we take \( \theta = 0 \) and substitute the inhomogeneous norms by the homogeneous ones, i.e. \( B^{-\sigma}_{r,k} \rightarrow B^{-\sigma}_{r,k}, B^0_{2,k} \rightarrow B^0_{2,k} \), then the estimates (2.2) and (2.4) satisfy scaling invariance.

**Proof of Lemma 4**

We concentrate on the case \( k = 1 \). From duality argument, it is sufficient to prove that
\[
\left\| \int_{-\infty}^{\infty} U(-t')F(t')dt' \right\|_{B_{r,\infty}^0} \lesssim \|F\|_{L^q_t B^\sigma_{r,k}},
\]
(2.5)
where \( q' \) and \( r' \) denote the Hölder conjugate of \( q \) and \( r \) respectively. In fact in [7] we find
\[
\left\| \int_{-\infty}^{\infty} U(-t')\varphi_k * F(t')dt' \right\|_{L^2} \lesssim 2^{k\sigma} \|\varphi_k * F\|_{L^q_t L^{r'}},
\]
(2.6)
where \( \{\varphi_k\}_{0}^{\infty} \) is the Littlewood–Paley dyadic decomposition on \( \mathbb{R}^n \) and \( q, r, \sigma \) are as in Lemma 4. We take supremum of \( k \) on both sides to obtain (2.5).

We use the following Gagliardo–Nirenberg type interpolation inequality ([3] or see [8] for more general cases).

**Lemma 5** *The following estimate holds.*
\[
\|f\|_{L^\infty} \lesssim \|f\|_{H^\alpha_p}^{\delta} \|f\|_{H^\beta_q}^{1-\delta},
\]
(2.7)
where \( 1 \leq p, q \leq \infty, 0 < \alpha, \beta < \infty, 0 < \delta < 1, \alpha > n/p, \beta > n/q, \delta(\alpha - n/p) + (1 - \delta)(\beta - n/q) = 0.\)

**Proof of Theorem 3**

We define the complete metric space \( \Phi = \Phi(p, s, k, M) \) for \( NLD \) as
\[
\Phi = \{ \psi \in L^\infty(\mathbb{R}; B^0_{2,k}) \cap L^{p-1}(\mathbb{R}; L^\infty); \|\psi\|_{L^p_t B^0_{2,k}} + \|\psi\|_{L^{p-1}_t L^\infty} \leq M \}.
\]
(2.8)
We find a unique solution of \( NLD \) in \( \Phi \) for sufficiently small data \( \psi \) and \( M \). For any \( s \in \mathbb{R}, k = 1, 2 \) and \( q, r, \sigma \) satisfying the condition in Lemma 4, we have from (2.1), (2.2) and (2.4),
\[
\|\psi\|_{L^q_t B^\sigma_{r,k}} \lesssim \|\psi\|_{B^0_{2,k}} + \|f(\psi)\|_{L^1_t B^0_{2,k}} \\
\lesssim \|\psi\|_{B^0_{2,k}} + \|\psi\|_{L^{p-1}_t L^\infty} \|\psi\|_{L^\infty_t B^0_{2,k}}.
\]
(2.9)
So we concentrate on the $L_t^p L^\infty$ norm. We apply Lemma 4 to estimate it in $L_t^q B^s_{r,k}$. 

(1) $n = 1$. For any $p \geq 5$, we take $k = 1$ and 

$$(s, q, r, \sigma, \theta) = (1/2 + 1/(p-1), p-1, \infty, 1/2 + 1/(p-1), 4/(p-1))$$

and use $B^0_{\infty, 1} \hookrightarrow L^\infty$ to obtain the theorem.

(2) $n = 2$. For any $3 < p \leq 5$, we take $k = 1$ and 

$$(s, q, r, \sigma, \theta) = (1/2 + 1/(p-1), p-1, \infty, 1/2 + 1/(p-1), (5-p)/(p-1))$$

and use $B^0_{\infty, 1} \hookrightarrow L^\infty$ to obtain the theorem.

(3) $n = 2$. For any $p > 5$, we take $k = 2$ and $0 < \delta < 1$ satisfying $(1-\delta)(p-1) \geq 4$. From (2.7), we estimate 

$$
\|\psi\|_{L_t^{p-1} L^\infty} \lesssim \|\psi\|_{L_t^{\infty} H^s} \|\psi\|_{L_t^{(1-\delta)(p-1)} H^{s-\sigma}}^{1-\delta}.
$$

We take 

$$(s, q, r, \sigma, \theta) = (1-1/(p-1), (1-\delta)(p-1), 1/2 -2/(1-\delta)(p-1), 3/(1-\delta)(p-1), 0)$$

(2.13) to obtain the theorem.

(6) $n \geq 4$, $p = 3$. We take $k = 1$ and 

$$(s, q, r, \sigma, \theta) = ((n-1)/2, 2, 2(n-1)/(n-3), (n+1)/2(n-1), 0)$$

(2.14) and use $B^s_{\infty, 1} \hookrightarrow L^\infty$ to obtain the theorem.

(7) $n \geq 4$. For any $p > 3$, we take $k = 2$ and $0 < \delta < 1$ satisfying $(1-\delta)(p-1) \geq 2$. From (2.7), we estimate (2.12) and take 

$$(s, q, r, \sigma, \theta) = (n/2 - 1/(p-1), (1-\delta)(p-1), (1/2-2/(1-\delta)(p-1)(n-1))^{-1},$$

$$(n+1)/(1-\delta)(p-1)(n-1), 0)$$

(3.1) to obtain the theorem.

### 3 Application for nonlinear Klein–Gordon equations

We apply the previous argument for the Klein–Gordon equation with derivative coupling ($NLKG$): 

$$
\partial_t^2 u - \Delta u + m^2 u = \lambda f(u), \quad u(0) = u_0, \quad \partial_t u(0) = u_1,
$$

where $u : \mathbb{R}^{1+n} \rightarrow \mathbb{C}$ is unknown, $u_0, u_1 : \mathbb{R}^n \rightarrow \mathbb{C}$ are given Cauchy data, $m > 0$ and $\lambda \in \mathbb{C}$ are constants. We consider the nonlinear term $f(u)$ of following types:

$$
f(u) = \partial_j(u^p), \partial_k(u^p), \prod_{j=1}^n (\partial_j u)^{p_j}, (\partial_t u)^{p_0} \prod_{j=1}^n (\partial_j u)^{p_j},
$$

(3.2)
where $1 \leq j \leq n$.

We give the results only. For simplicity we set $\varphi = (u_0, u_1)$ and $\|\varphi\|_{B_{2,k}^k} := \|u_0\|_{B_{2,k}^k} + \|u_1\|_{B_{2,k}^{s-1}}$.

**Theorem 6** Let $n, p, \phi$ satisfy the following conditions:

1. $n = 1, \quad p \geq 5, \quad \|\varphi\|_{B_{2,1}^{s+1}} \ll 1, \quad s = 1/2 + 1/(p - 1),$
2. $n = 2, \quad 3 < p \leq 5, \quad \|\varphi\|_{B_{2,1}^{s+1}} \ll 1, \quad s = 1/2 + 1/(p - 1),$
3. $n = 2, \quad p > 5, \quad \|\varphi\|_{H^{s(p)}_{1}} \ll 1, \quad s > 1,$
4. $n = 3, \quad p = 3, \quad \|\varphi\|_{H^{s(p)}_{1}} \ll 1, \quad s > 1,$
5. $n \geq 4, \quad p = 3, \quad \|\varphi\|_{B_{2,1}^{3(s)}} \ll 1,$
6. $n \geq 4, \quad p > 3, \quad \|\varphi\|_{H^{s(p)}_{1}} \ll 1, \quad (s(p) < p \text{ if } p \neq \text{ integer}),$

(1) – (7) for $f = \partial_j (w^p), \quad (4) – (7)$ for $f = \partial_t (u^p)$.

Then NLKG has a solution $\psi \in C(\mathbb{R}; X)$, where $X$ denotes the space of data $\varphi$ indicated above.

**Remark** In this case $s(p)$ is scaling critical exponent for massless NLKG ($m = 0$).

**Theorem 7** Let $n, p, \phi$ satisfy the following conditions:

1. $n = 1, \quad p \geq 5, \quad \|\varphi\|_{B_{2,1}^{s+1}} \ll 1, \quad s = 1/2 + 1/(p - 1),$
2. $n = 2, \quad 3 < p \leq 5, \quad \|\varphi\|_{B_{2,1}^{s+1}} \ll 1, \quad s = 1/2 + 1/(p - 1),$
3. $n = 2, \quad p > 5, \quad \|\varphi\|_{H^{s(p)}_{1}} \ll 1, \quad s > 1,$
4. $n = 3, \quad p = 3, \quad \|\varphi\|_{H^{s(p)}_{1}} \ll 1, \quad s > 1,$
5. $n \geq 4, \quad p = 3, \quad \|\varphi\|_{B_{2,1}^{3(s)}} \ll 1,$
6. $n \geq 4, \quad p > 3, \quad \|\varphi\|_{H^{s(p)}_{1}} \ll 1,$

(1) – (7) for $f = \prod_{j=1}^n (\partial_j u)^{p_j}, \quad p_1 + \cdots + p_n = p, \quad p_j \in \mathbb{Z}^+ \cup \{0\} \text{ or } p_j > \max\{1, s\},$

(4) – (7) for $f = (\partial_t u)^{p_0} \prod_{j=1}^n (\partial_j u)^{p_j}, \quad p_0 + \cdots + p_n = p, \quad p_j \in \mathbb{Z}^+ \cup \{0\} \text{ or } p_j > \max\{1, s\}.$

Then NLKG has a solution $\psi \in C(\mathbb{R}; X)$, where $X$ denotes the space of data $\varphi$ indicated above.

**Remark** In this case $s(p) + 1$ is scaling critical exponent for massless NLKG ($m = 0$).

**References**


