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Kyoto University
Effective mass and mass renormalization of nonrelativistic QED

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Abstract

The effective mass $m_{\text{eff}}$ of the nonrelativistic QED is considered. $m_{\text{eff}}$ is defined as the inverse of curvature of the ground state energy with total momentum zero. The effective mass $m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)$ is a function of bare mass $m > 0$, ultraviolet cutoff $\Lambda > 0$, infrared cutoff $\kappa > 0$, and the square of charge $e$ of an electron. Introduce a scaling $m \to m(\Lambda) = (b\Lambda)^\beta$, $\beta < 0$. Then asymptotics behavior of $m_{\text{eff}}$ as $\Lambda \to \infty$ is studied.

1 Introduction

1.1 The Pauli-Fierz Hamiltonian

This is a joint work with Herbert Spohn.¹ We consider a single, spinless free electron coupled to a quantized radiation field (photons). The Hilbert space of states of photons is the symmetric Fock space:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left( \otimes_{s}^{n} L^2(\mathbb{R}^3 \times \{1, 2\}) \right),$$

where $\otimes_{s}^{n} L^2(\mathbb{R}^3 \times \{1, 2\})$ denotes the $n$-fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1, 2\})$ with $\otimes_{s}^{0} L^2(\mathbb{R}^3 \times \{1, 2\}) = \mathbb{C}$. The inner product in $\mathcal{F}$ is denoted by $(\cdot, \cdot)$ and the Fock vacuum by $\Omega$. On $\mathcal{F}$ we introduce the Bose field

$$a(f) = \sum_{j=1,2} \int f(k, j)^* a(k, j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1, 2\}),$$

(1.1)

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where \( a(f) \) and \( a^*(f) = a(\bar{f})* \) are densely defined and satisfy the CCR

\[
\begin{align*}
[a(f), a^*(g)] &= (f, g)_{L^2(\mathbb{R}^3 \times \{1, 2\})}, \\
[a(f), a(g)] &= 0, \\
[a^*(f), a^*(g)] &= 0.
\end{align*}
\]

The free Hamiltonian of \( \mathcal{F} \) is read as

\[
H_f = \sum_{j=1,2} \int \omega(k)a^*(k,j)a(k,j)dk,
\]

(1.2)

where the dispersion relation is given by

\[ \omega(k) = |k|. \]

The free Hamiltonian \( H_f \) acts as

\[
H_f \Omega = 0, \\
H_f a^*(f_1) \cdots a^*(f_n) \Omega = \sum_{j=1}^{n} a^*(f_1) \cdots a^*(\omega f_j) \cdots a^*(f_n) \Omega.
\]

The Pauli-Fierz Hamiltonian \( H \) is defined as a self-adjoint operator acting on

\[ \mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^\oplus \mathcal{F} dx \]

by

\[ H = \frac{1}{2m}(p_x \otimes 1 - eA_\varphi)^2 + V \otimes 1 + 1 \otimes H_f, \]

where \( m \) and \( e \) denote the mass and charge of electron, respectively,

\[ p_x = \left( -i \frac{\partial}{\partial x_1}, -i \frac{\partial}{\partial x_2}, -i \frac{\partial}{\partial x_3} \right) \]

and \( V \) an external potential. The quantized radiation field \( A_\varphi \) is defined by

\[
A_\varphi = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3}^\oplus (a(f_x) + a^*(\bar{f}_x)) dx,
\]

(1.3)

where

\[
f_x(k,j) = \frac{1}{\sqrt{\omega}} \varphi(k) e(k,j) e^{i k_x},
\]

(1.4)

\( e(k,1), e(k,2), k/|k| \) form a right-handed dreibain, and \( \varphi \) is a form factor. \( A_\varphi \) acts for \( \Psi \in \mathcal{H} \) as

\[
(A_\varphi \Psi)(x) = (a(f_x) + a^*(\bar{f}_x)) \Psi(x), \quad x \in \mathbb{R}^3.
\]

**Theorem 1.1** Assume that \( \varphi/\sqrt{\omega}, \varphi/\sqrt{\omega}, \varphi/\sqrt{\omega} \in L^2(\mathbb{R}^3) \) and \( V \) is relatively bounded with respect to \(-\Delta\) with a relative bound \(<1\). Then, for arbitrary values of \( e \), \( H \) is self-adjoint on \( D(\Delta \otimes 1) \cap D(1 \otimes H_f) \) and bounded from below.

**Proof:** See Hiroshima [3, 4].
1.2 Effective mass

The momentum of the photon field is given by

\[ P_\text{f} = \sum_{j=1,2} \int ka^*(k,j)a(k,j)dk \]  

(1.5)

and the total moment by

\[ P_\text{total} = p_z \otimes 1 + 1 \otimes P_\text{f} \]

Let us assume that 

\[ V \equiv 0. \]

Then we see that 

\[ [H, P_\text{total}_\mu] = 0, \quad \mu = 1, 2, 3. \]

Hence \( H \) and \( \mathcal{H} \) can be decomposable with respect to \( \text{Spec}(P_\text{total}) = \mathbb{R}^3 \), i.e.,

\[ \mathcal{H} = \int_{\mathbb{R}^3} \mathcal{H}(p)dp, \]

\[ H = \int_{\mathbb{R}^3} H(p)dp. \]

Note that

\[ e^{-iz\otimes P_\text{f}}P_\text{total}e^{iz\otimes P_\text{f}} = p_z, \]

\[ e^{-iz\otimes P_\text{f}}He^{iz\otimes P_\text{f}} = \frac{1}{2m}(p_z \otimes 1 - 1 \otimes P_\text{f} - e1 \otimes A_\varphi(0)) + 1 \otimes H_\text{f}, \]

where

\[ A_\varphi(0) = \frac{1}{\sqrt{2}}(a(f_0) + a(\overline{f}_0)). \]

From this we obtain that for each \( p \in \mathbb{R}^3 \),

\[ \mathcal{H}(p) \cong \mathcal{F}, \]

\[ H(p) \cong \frac{1}{2m}(p - P_\text{f} - eA_\varphi(0)) + H_\text{f}, \]

Let

\[ E_{m,\Lambda}(p) = \inf \text{Spec}(H(p)). \]  

(1.6)

Let us assume sharp ultraviolet cutoff \( \Lambda \) and infrared cutoff \( \kappa \), which means

\[ \varphi(k) = \begin{cases} 0 & \text{for } |k| < \kappa, \\ (2\pi)^{-3/2} & \text{for } \kappa \leq |k| \leq \Lambda, \\ 0 & \text{for } |k| > \Lambda. \end{cases} \]  

(1.7)

**Lemma 1.2** There exists constants \( p_* \) and \( e_* \) such that for

\[ (p, e) \in \mathcal{O} = \{(p, e) \in \mathbb{R}^3 \times \mathbb{R}||p| < p_*, |e| < e_*\}, \]

\( H(p) \) has a ground state \( \psi_\text{g}(p) \) and it is unique. Moreover \( \psi_\text{g}(p) = \psi_\text{g}(p, e) \) is strongly analytic and \( E_{m,\Lambda}(p) = E_{m,\Lambda}(p, e) \) analytic with respect to \( (p, e) \in \mathcal{O}. \).
Proof: See Hiroshima and Spohn [6, 7].

In what follows we assume that \((p, e) \in \mathcal{O}\).

**Definition 1.3** The effective mass \(m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)\) is defined by

\[
\frac{1}{m_{\text{eff}}} = \frac{1}{3} \Delta_{p} E(p, e)|_{p=0}.
\]

### 1.3 Mass renormalization

Removal of the ultraviolet cutoff \(\Lambda\) through mass renormalization means to find sequences

\[
\Lambda \to \infty, \quad m \to 0
\]

such that \(E_{m, \Lambda}(p) - E_{m, \Lambda}(0)\) has a nondegenerate limit. To achieve this, as a first step we want to find constants

\[
\beta < 0, \quad 0 < b
\]

such that

\[
\lim_{\Lambda \to \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b \Lambda)^\beta) = m_{\text{ph}},
\]

where \(m_{\text{ph}}\) is a given constant. Actually \(m_{\text{ph}}\) is a physical mass. Namely in the mass renormalization the scaled bare mass goes to zero and the effective mass goes to a physical mass as the ultraviolet cutoff \(\Lambda\) goes to infinity.

We will see later that \(m_{\text{eff}}/m\) is a function of \(e^2, \Lambda/m\) and \(\kappa/m\). Let

\[
\frac{m_{\text{eff}}}{m} = f(e^2, \Lambda/m, \kappa/m),
\]

where \(f(0, \Lambda/m, \kappa/m) = 1\) holds. An analysis of (1.10) can be reduced to investigate the asymptotic behavior of \(f\) as \(\Lambda \to \infty\). Namely we want to find constants

\[
0 \leq \gamma < 1, \quad 0 < b_0
\]

such that

\[
\lim_{\Lambda \to \infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)^\gamma} = b_0.
\]

If we succeed to find constants \(\gamma\) and \(b_0\) such as in (1.12) then by

\[
m_{\text{eff}}(e^2, \Lambda, \kappa, m) = mf(e^2, \Lambda/m, \kappa/m),
\]

we have

\[
m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b \Lambda)^\beta) = (b \Lambda)^\beta f(e^2, \Lambda/(b \Lambda)^\beta, \kappa/(b \beta)) \approx b_0 (b \Lambda)^\beta (\Lambda/(b \Lambda)^\beta)^\gamma.
\]

Taking

\[
\beta = \frac{-\gamma}{1 - \gamma} < 0, \quad b = 1/b_1^{1/\gamma},
\]

we have

\[
m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b \Lambda)^\beta) \approx b_0 (b \Lambda)^\beta (\Lambda/(b \Lambda)^\beta)^\gamma.
\]
we see that by (1.13)

$$\lim_{\Lambda \to \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b \Lambda)^\beta) = \lim_{\Lambda \to \infty} b_0 \left( \frac{\Lambda}{b_1^{1/\gamma}} \right) \left( \frac{\Lambda}{(\Lambda/(b_1)^{1/\gamma})^\beta} \right)^\gamma = b_0 b_1,$$

where $b_1$ is a parameter, which is adjusted such as

$$b_0 b_1 = m_{\text{ph}}.$$

Hence we will be able to establish (1.10). It is easily seen that

$$f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log \left( \frac{\Lambda/m + 2}{\kappa/m + 2} \right) + O(\alpha^2),$$

where $\alpha = e^2/4\pi$, which suggests

$$f(e^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{8\alpha/3\pi},$$

for sufficiently small $\alpha$ and large $\Lambda$, and therefore

$$\gamma = 8\alpha/3\pi.$$

One may assume that

$$f(e^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{\alpha(8/3\pi) + \alpha^2 b}$$

for sufficiently small $\alpha$ with some constant $b$. Then by expanding $m_{\text{eff}}/m$ to order $\alpha^2$ one may expect that

$$f(e^2, \Lambda/m, \kappa/m) \approx 1 + \alpha \frac{8}{3\pi} \log \left( \frac{\Lambda}{m} \right) + \frac{1}{2} \alpha^2 \left( \frac{8}{3\pi} \log \left( \frac{\Lambda}{m} \right) \right)^2 + b \alpha^2 \log \left( \frac{\Lambda}{m} \right) + O(\alpha^3)$$

(1.14)

for sufficiently small $\alpha$ and large $\Lambda$. It is, however, that (1.14) is not confirmed. Instead of (1.14) we prove that there exists a constant $C > 0$ such that

$$f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log \left( \frac{\Lambda/m + 2}{\kappa/m + 2} \right) + \alpha^2 C \sqrt{\Lambda/m} + O(\alpha^3).$$

The effective mass and its renormalization have been studied from a mathematical point of view by many authors. Spohn [10] investigates the effective mass of the Nelson model [9] from a functional integral point of view. Lieb and Loss [8] studied mass renormalization and binding energies of models of matter coupled to radiation fields including the Pauli-Fierz model. Hainzl and Seiringer [2] computed exactly the leading order in $\alpha$ of the effective mass of the Pauli-Fierz Hamiltonian with spin.
2 Perturbative expansions

The effective masses for $H(p)$ and

$$\frac{1}{2m} : (p - P_t - eA_{\phi}(0))^2 : + H_t$$

are identical. Then in what follows we redefine $H(p)$ as

$$H(p) = \frac{1}{2m} : (p - P_t - eA_{\phi}(0))^2 : + H_t.$$ 

Furthermore for notational convenience we write $A$ and $E(p)$ for $A_{\phi}(0)$ and $E_{m,\Lambda}(p)$, respectively.

2.1 Formulae

Lemma 2.1 We have

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} (\psi_g(0), (P_f + eA)_{\mu} (H(0) - E(0))^{-1} (P_f + eA)_{\mu} \psi(0)) (\psi_g(0), \psi_g(0)).$$

Proof: It is seen that $E(p, e) = E(p, -e) = E(-p, e)$. Then

$$\frac{\partial}{\partial p_{\mu}} E(p, e) \bigg|_{p_{\mu}=0} = 0, \quad \mu = 1, 2, 3,$$

follows. Moreover it is seen that $E(p, e)$ is a function of $e^2$ and

$$\frac{d^{2m-1}}{de^{2m-1}} E(p, e) \bigg|_{e=0} = 0.$$ 

In this proof, $f'(p)_{\mu}$ means the strong derivative of $f(p)$ with respect to $p_{\mu}$. Since

$$H(p) \psi_g(p) = E(p) \psi_g(p),$$

we have

$$H'(p)_{\mu} \psi_g(p) + H(p) \psi'_g(p)_{\mu} = E'(p)_{\mu} \psi_g(p) + E(p) \psi'_g(p)_{\mu}$$

and

$$H''(p)_{\mu} \psi_g(p) + 2H'(p)_{\mu} \psi'_g(p)_{\mu} + H(p) \psi''_g(p)_{\mu} = E''(p)_{\mu} \psi_g(p) + 2E'(p)_{\mu} \psi'_g(p)_{\mu} + E(p) \psi''_g(p)_{\mu}.$$ 

By (2.1) it follows that $E'(0)_{\mu} = 0$, and by (2.3) with $p = 0$,

$$(P_t + eA)_{\mu} \psi_g(0) \in D((H(0) - E(0))^{-1}),$$

$$\psi'_g(0)_{\mu} = (H(0) - E(0))^{-1} (P_t + eA)_{\mu} \psi_g(0).$$
Then we have by (2.3) and (2.4),
\[
\frac{m}{m_{\text{eff}}} = \frac{1}{3} \sum_{\mu=1,2,3} \frac{(\psi_{g}(0), E''(0)_{\mu} \psi_{g}(0))}{(\psi_{g}(0), \psi_{g}(0))}
\]
\[
= 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{(P_{I} + eA)_{\mu} \psi_{g}(0), (H(0) - E(0))^{-1}(P_{I} + eA)_{\mu} \psi_{g}(0))}{(\psi_{g}(0), \psi_{g}(0))}
\]
Thus the lemma follows. \(\square\)

Let
\[
\psi_{g}(0) = \sum_{n=0}^{\infty} \frac{e^{n}}{n!} \varphi_{n}, \quad E(0) = \sum_{n=0}^{\infty} \frac{e^{2n}}{(2n)!} E_{2n}.
\]
Note that
\[
\varphi_{2m} \in \bigoplus_{m=0}^{\infty} F^{(2m)}, \quad \varphi_{2m+1} \in \bigoplus_{m=0}^{\infty} F^{(2m+1)}.
\]
We want to get the explicit form of \(\varphi_{n}\). Let
\[
F_{\text{fin}} = \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in F|\Psi^{(m)} = 0 \text{ for } m \geq \ell \text{ with some } \ell \right\},
\]
\[
F_{0} = \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in F_{\text{fin}} \right| \begin{array}{l}
(i) \Psi^{(0)} = 0, \\
(ii) \text{supp}(k_{1}, \ldots, k_{n}) \in R^{3n} \Psi^{(n)}(k_{1}, \ldots, k_{n}, j_{1}, \ldots, j_{n}) \neq \{(0, \ldots, 0)\}.
\end{array}
\]

Lemma 2.2 We see that \(F_{0} \subset D(H_{0}^{-1})\).

**Proof:** Let \(\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in F_{0}\). Since
\[
(H_{0} \Psi)^{(n)}(k_{1}, \ldots, k_{n}, j_{1}, \ldots, j_{n})
\]
\[
= \left[ \frac{1}{2} (k_{1} + \cdots + k_{n})^{2} + \sum_{j=1}^{n} \omega(k_{j}) \right] \Psi^{(n)}(k_{1}, \ldots, k_{n}, j_{1}, \ldots, j_{n}),
\]
we see that
\[
(H_{0}^{-1} \Psi)^{(n)}(k_{1}, \ldots, k_{n}, j_{1}, \ldots, j_{n})
\]
\[
= \left[ \frac{1}{2} (k_{1} + \cdots + k_{n})^{2} + \sum_{j=1}^{n} \omega(k_{j}) \right]^{-1} \Psi^{(n)}(k_{1}, \ldots, k_{n}, j_{1}, \ldots, j_{n}).
\]
Since \(\text{supp}(k_{1}, \ldots, k_{n}) \in R^{3n} \Psi^{(n)}(k_{1}, \ldots, k_{n}, j_{1}, \ldots, j_{n}) \neq \{(0, \ldots, 0)\}\), we obtain that
\[
\|H_{0}^{-1} \Psi\|_{F}^{2} = \sum_{n=1}^{\text{finite}} \|\Psi^{(n)}\|_{F}^{2} < \infty.
\]
Then the lemma follows. \(\square\)

We split \(H(0)\) as
\[
H(0) = H_{0} + eH_{1} + \frac{e^{2}}{2} H_{2},
\]
where
\[
H_0 = \frac{1}{2} P_t^2 + H_t, \\
H_1 = \frac{1}{2} (P_t \cdot A + A \cdot P_t) = P_t \cdot A = A \cdot P_t, \\
H_2 = A^2.
\]

Lemma 2.3 We have \(E_0 = E_1 = E_2 = E_3 = 0\) and
\[
\varphi_0 = \Omega, \quad \varphi_1 = 0, \quad \varphi_2 = -H_0^{-1}H_2\Omega, \quad \varphi_3 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega.
\]
In particular \(\varphi_2 \in \mathcal{F}^{(2)}\) and \(\varphi_3 \in \mathcal{F}^{(1)} \cap \mathcal{F}^{(3)}\).

Proof: Let us set \(H(0), E(0)\) and \(\psi_g(0)\) as \(H, E\) and \(\psi_g\), respectively. It is obvious that \(E_0 = 0\) and \(\varphi_0 = a\Omega\) with arbitrary \(a \in \mathbb{C}\), and by (2.2), \(E_1 = E_3 = 0\). Set \(a = 1\). We denote the strong derivative of \(f = f(e)\) with respect to \(e\) by \(f'\). We have
\[
H'\psi_g + H\psi_g' = E'\psi_g + E\psi_g' \tag{2.5}
\]
and
\[
H''\psi_g + 3H'\psi_g' + H\psi_g'' = E''\psi_g + 3E'\psi_g' + E\psi_g'' \tag{2.6}
\]
From (2.6) it follows that
\[
(\psi_g, H''\psi_g) + (\psi_g, 2H'\psi_g') + (\psi_g, H\psi_g'') = E''(\psi_g, \psi_g) + (\psi_g, 2E'\psi_g') + (\psi_g, E\psi_g''). \tag{2.7}
\]
Put \(e = 0\) in (2.7). Then
\[
(\Omega, H_2\Omega) + (\Omega, 2H_1\Omega) + (\Omega, H_0\varphi_2) = E_2(\Omega, \Omega). \tag{2.8}
\]
Since the left-hand side of (2.8) vanishes, we have \(E_2 = 0\). From (2.5) with \(e = 0\) and the fact \(E_0 = E_1 = 0\), it follows that
\[
H_1\Omega + H_0\varphi_1 = 0,
\]
from which it holds that \(H_0\varphi_1 = 0\). Since \(H_0\) has the unique eigenvector \(\Omega\) (the ground state) with eigenvalue zero, it follows that \(\varphi_1 = b\Omega\) with some constant \(b\). \(\varphi_1 \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m+1)}\) which implies \(b = 0\). Hence \(\varphi_1 = 0\) follows. By (2.6) with \(e = 0\), we have
\[
H_2\Omega + 2H_1\varphi_1 + H_0\varphi_2 = 0.
\]
Since \(H_2\Omega \in \mathcal{F}_0\), we see that by Lemma 2.2, \(H_2\Omega \in D(H_0^{-1})\). Thus we have \(\varphi_2 = -H_0^{-1}H_2\Omega\). From the identity
\[
H''\psi_g + 3H'\psi_g' + H\psi_g'' = E''\psi_g + 3E'\psi_g' + E\psi_g'' \tag{2.9}
\]
it follows that at \(e = 0\),
\[
3H_1\varphi_2 + H_0\varphi_3 = 0.
\]
Since \(H_1\varphi_2 = -H_1H_0^{-1}H_2\Omega \in \mathcal{F}_0\), Lemma 2.2 ensures that \(H_1\varphi_2 \in D(H_0^{-1})\). Hence \(\varphi_3 = -3H_0^{-1}H_1\varphi_2 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega\). Then the lemma is proven.
\(\square\)
2.2 Order $e^4$

In this subsection we expand $m/m_{\text{eff}}$ up to order $e^4$. We define $A^-$ and $A^+$ by

$$A^- = \frac{1}{\sqrt{2}} a(f), \quad A^+ = \frac{1}{\sqrt{2}} a^*(f).$$

Then $A = A^+ + A^-.$

**Lemma 2.4** We have

$$\frac{m}{m_{\text{eff}}} = 1 - e^2 \frac{2}{3} \sum_{\mu=1}^{3} \left( \Omega, A_{\mu} H_{0}^{-1} A_{\mu} \Omega \right)$$

$$-e^4 \frac{2}{3} \sum_{\mu=1}^{3} \left\{ 2 \left( \Psi_3^\mu, H_{0}^{-1} \Psi_1^\mu \right) + \left( \Psi_2^\mu, H_{0}^{-1} \Psi_2^\mu \right) - 2 \left( \Psi_2^\mu, H_{0}^{-1} H_{1} H_{0}^{-1} \Psi_1^\mu \right) \\ - \frac{1}{2} \left( \Psi_1^\mu, H_{0}^{-1} H_{2} H_{0}^{-1} \Psi_1^\mu \right) + \left( \Psi_1^\mu, H_{0}^{-1} H_{1} H_{0}^{-1} H_{1} H_{0}^{-1} \Psi_1^\mu \right) \right\} + O(e^6),$$

(2.10)

where

$$\Psi_1^\mu = A_{\mu} \Omega,$$

$$\Psi_2^\mu = -\frac{1}{2} P_{f} H_{0}^{-1} (A^+ \cdot A^+) \Omega,$$

$$\Psi_3^\mu = \frac{1}{2} \left\{ -A_{\mu} H_{0}^{-1} (A^+ \cdot A^+) \Omega + \frac{1}{2} P_{f} H_{0}^{-1} (P_{f} \cdot A + A \cdot P_{f}) H_{0}^{-1} (A^+ \cdot A^+) \Omega \right\}.$$

**Proof:** In Lemma 2.1 we have seen that

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{\left( (P_f + eA)_{\mu} \psi_{g}(0), (H(0) - E(0))^{-1} (P_f + eA)_{\mu} \psi_{g}(0) \right)}{\left( \psi_{g}(0), \psi_{g}(0) \right)}. $$

(2.11)

We can strongly expand $(H(0) - E(0))^{-1}$ as

$$(H(0) - E(0))^{-1} = H_{0}^{-1} - e H_{0}^{-1} H_{1} H_{0}^{-1}$$

$$+ e^2 \left( -\frac{1}{2} H_{0}^{-1} H_{2} H_{0}^{-1} + H_{0}^{-1} H_{1} H_{0}^{-1} H_{1} H_{0}^{-1} \right) + O(e^3).$$

(2.12)

Here we set

$$H_j = \begin{cases} H_j, & j = 1, 2, \\
- E_j, & j \geq 3. \end{cases}$$

Note that

$$\varphi_0 \in \mathcal{F}^{(0)}, \varphi_2 \in \mathcal{F}^{(2)}, \varphi_3 \in \mathcal{F}^{(3)} \cap \mathcal{F}^{(1)}, \varphi_4 \in \mathcal{F}^{(4)} \cap \mathcal{F}^{(2)}.$$
In particular
\[
\frac{1}{(\psi_g, \psi_g)} = 1 - e^4 \left( \frac{1}{2} \varphi_2, \frac{1}{2} \varphi_2 \right) - e^4 \left( \Omega, \frac{1}{24} \varphi_4 \right) + O(e^6) = 1 - e^4 \frac{1}{4} \left( \varphi_2, \varphi_2 \right) + O(e^6).
\]

Moreover we have
\[
(P_f + eA)_{\mu} \psi_g(0) = eA_{\mu} \Omega + e^2 \left( \frac{1}{2} P_{f\mu} \varphi_2 \right) + e^3 \left( \frac{1}{2} A_{\mu} \varphi_2 + \frac{1}{6} P_{f\mu} \varphi_3 \right) + O(e^4)
= e\Psi_1^\mu + e^2 \Psi_2^\mu + e^3 \Psi_3^\mu + O(e^4).
\]
(2.14)

Substitute (2.12), (2.13) and (2.14) into (2.11). Then the lemma follows. \[\square\]

For each \( k \in \mathbb{R}^3 \) let us define the projection \( Q(k) \) on \( \mathbb{R}^3 \) by
\[
Q(k) = \sum_{j=1,2} |e_j(k)\rangle\langle e_j(k)|.
\]

We set
\[
\varphi_j = \varphi(k_j), \quad \omega_j = \omega(k_j), \quad Q(k_j) = Q_j, \quad j = 1, 2.
\]

Let
\[
\frac{1}{F_j} = \frac{1}{r_j^2/2 + r_j^2}, \quad j = 1, 2,
\]
\[
\frac{1}{F_{12}} = \frac{1}{(r_1^2 + 2r_1r_2X + r_2^2)/2 + r_1 + r_2}, \quad r_1, r_2 \geq 0, \quad -1 \leq X \leq 1.
\]

**Lemma 2.5** We have
\[
\frac{m}{m_{\text{eff}}} = 1 - a_1(\Lambda/m, \kappa/m) - a_2(\Lambda/m, \kappa/m) + O(a^3),
\]
where
\[
a_1(\Lambda/m, \kappa/m) = \frac{8}{3\pi} \log \left( \frac{\Lambda/m + 2}{\kappa/m + 2} \right)
\]
and
\[
a_2(\Lambda/m, \kappa/m) = \frac{(4\pi)^2}{(2\pi)^6} \frac{2}{3} \int_{-1}^1 dX \int_{\kappa/m}^{\Lambda/m} dr_1 \int_{\kappa/m}^{\Lambda/m} dr_2 \pi r_1 r_2 \times
\]
\[
\times \left\{ - \left( \frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}} \left( 1 + X^2 \right) + \left( \frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1r_2X + r_2^2}{2} (1 + X^2) \right.
\]
\[
+ \left( \frac{1}{F_1} + \frac{1}{F_2} \right) \left( \frac{1}{F_{12}} \right)^2 r_1r_2X(-1 + X^2) - \frac{1}{F_1 F_2} (1 + X^2)
\]
\[
+ \left( \frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2} \right) \frac{1}{F_{12}} (1 - X^2) + \frac{1}{F_1 F_2 F_{12}} r_1r_2X(-1 + X^2) \right\}.
\]
(2.16)
Proof: Note that

\[ a_1(\Lambda, \kappa) = \frac{2}{3}(\sqrt{4\pi})^2(A^+_\mu\Omega, H_0^{-1}A^+_\mu\Omega) \]

\[ = \frac{8}{3\pi}\log\left(\frac{\Lambda/m+2}{\kappa/m+2}\right). \]

Thus (2.15) follows. To see \( a_2(\Lambda, \kappa) \) we exactly compute the five terms on the right-hand side of (2.10) separately. Let

\[ \frac{1}{E_j} = \frac{1}{|k_j|^2/2 + \omega_j}, \quad j = 1, 2, \]

\[ \frac{1}{E_{12}} = \frac{1}{|k_1 + k_2|^2/2 + \omega_1 + \omega_2}. \]

(1) We have

\[ 2(\Psi_3^\mu, H_0^{-1}\Psi_3^\mu) = (\Omega, -(A^-\cdot A^-)H_0^{-1}A_\mu H_0^{-1}A^+_\mu\Omega) \]

\[ + \frac{1}{2}(\Omega, (A^-\cdot A^-)H_0^{-1}(P_{\mathrm{f}} \cdot A + A \cdot P_{\mathrm{f}})H_0^{-1}P_{\mathrm{f}} H_0^{-1}A^+_\mu\Omega). \]

\[ = - \iint \frac{1}{E_1} \frac{1}{E_2} |\hat{\varphi}_1|^2 |\hat{\varphi}_2|^2 \frac{1}{E_{12}} (\frac{1}{E_1} + \frac{1}{E_2}) \mathrm{tr}(Q_1Q_2). \]  

(2.17)

(2) We have

\[ (\Psi_2^\mu, H_0^{-1}\Psi_2^\mu) = \left( \frac{1}{2} \right)^2 (P_{\mu} H_0^{-1}(A^+\cdot A^+)\Omega, H_0^{-1}P_{\mu} H_0^{-1}(A^+\cdot A^+)\Omega) \]

\[ = \left( \frac{1}{2} \right)^2 \iint \frac{1}{E_1} \frac{1}{E_2} |\hat{\varphi}_1|^2 |\hat{\varphi}_2|^2 \frac{1}{E_{12}} |k_1 + k_2|^2 2 \mathrm{tr}(Q_1Q_2). \]

(2.18)

(3) We have

\[ -2(\Psi_2^\mu, H_0^{-1}H_1 H_0^{-1}\Psi_1^\mu) \]

\[ = \left( \frac{1}{2} \right) (P_{\mu} H_0^{-1}(A^+\cdot A^+)\Omega, H_0^{-1}(P_{\mu} A + A P_{\mu}) H_0^{-1}A^+_\mu\Omega) \]

\[ = \iint \frac{1}{E_1} \frac{1}{E_2} |\hat{\varphi}_1|^2 |\hat{\varphi}_2|^2 \frac{1}{E_{12}} (\frac{1}{E_1} + \frac{1}{E_2})(k_2, Q_1 Q_2 k_1). \]  

(2.19)

(4) We have

\[ -\frac{1}{2}(\Psi_1^\mu, H_0^{-1}H_2 H_0^{-1}\Psi_1^\mu) \]

\[ = -\frac{1}{2} (A^+_\mu\Omega, H_0^{-1}((A^+\cdot A^+) + 2(A^+\cdot A^-) + (A^-\cdot A^-)) H_0^{-1}A^+_\mu\Omega) \]

\[ = - \iint \frac{1}{E_1} \frac{1}{E_2} |\hat{\varphi}_1|^2 |\hat{\varphi}_2|^2 \frac{1}{E_{12}} \mathrm{tr}(Q_1Q_2). \]  

(2.20)
We have
\[
\left( \Psi_1^\mu, H_0^{-1}H_1H_0^{-1}H_1H_0^{-1}\Psi_1^\mu \right)
= \left( \frac{1}{2} \right)^2 \left( A^+\Omega, H_0^{-1}(P \cdot A + A \cdot P)H_0^{-1}(P \cdot A + A \cdot P)H_0^{-1}A^+\Omega \right)
\]
\[
= \int \int dk_1^2 dk_2^2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_{12}} \left\{ \left( \frac{1}{E_1} \right)^2 (k_1, Q_2k_1) + \left( \frac{1}{E_2} \right)^2 (k_2, Q_1k_2) \right\}
+ \int \int \frac{1}{2\omega_1} \frac{1}{2\omega_2} \frac{1}{E_1} \frac{1}{E_2} (k_2, Q_1Q_2k_1).
\]  
(2.21)

Changing variables to the polar coordinate, we obtain (2.16) from Lemma 2.4, (2.17), (2.18), (2.19), (2.20), (2.21) and the facts
\[
\text{tr}[Q_1Q_2] = 1 + (\hat{k}_1, \hat{k}_2)^2,
\]
\[
(k_1, Q_2Q_1k_2) = (k_1, k_2)((\hat{k}_1, \hat{k}_2)^2 - 1),
\]
\[
(k_1, Q_2k_1) = |7c_1|^2(1 - (\hat{k}_1, \hat{k}_2)^2).
\]
Thus the proof is complete. \(\square\)

3 Main theorem

The main theorem is as follows.

**Theorem 3.1** There exist strictly positive constants \(C_{\min}\) and \(C_{\max}\) such that
\[
C_{\min} \leq \lim_{\Lambda \to \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.
\]

**Proof:** We show an outline of a proof. See Hiroshima and Spohn [7] for details.
By (2.16) we can see that
\[
a_2(\Lambda, \kappa) = \frac{(4\pi)^2}{(2\pi)^6} \frac{2}{3} \sum_{j=1}^{6} b_j(\Lambda/m), \tag{3.1}
\]
where
\[
b_1(\Lambda/m) = -\int (1 + X^2) \left( \frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}},
\]
\[
b_2(\Lambda/m) = \int (1 + X^2) \left( \frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1r_2X + r_2^2}{2},
\]
\[
b_3(\Lambda/m) = \int X(-1 + X^2)r_1r_2 \left( \frac{1}{F_1} + \frac{1}{F_2} \right) \left( \frac{1}{F_{12}} \right)^2,
\]
\[
b_4(\Lambda/m) = -\int (1 + X^2) \frac{1}{F_1} \frac{1}{F_2},
\]
\[
b_5(\Lambda/m) = \int (1 - X^2) \left( \frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2} \right) \frac{1}{F_{12}},
\]
\[
b_6(\Lambda/m) = \int X(-1 + X^2)r_1r_2 \frac{1}{F_1} \frac{1}{F_2} \frac{1}{F_{12}}.
\]
where
\[
\int = \int_{-1}^{1} \mathrm{d}X \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_1 \int_{\kappa/m}^{\Lambda/m} \mathrm{d}r_2 \pi r_1 r_2
\].

Let \(\rho_A(\cdot, \cdot) : [0, \infty) \times [-1, 1] \to \mathbb{R}\) be defined by
\[
\rho_A = \rho_A(r, X) = r^2 + 2\Lambda r X + \Lambda^2 + 2r + 2\Lambda = (r + \Lambda X + 1)^2 + \Delta,
\]
where
\[
\Delta = \Lambda^2(1 - X^2) + 2\Lambda(1 - X) - 1.
\]

Then we can show that there exist constants \(C_1, C_2, C_3\) and \(C_4\) such that for sufficiently large \(\Lambda > 0\),

1. \(\int_{-1}^{1} dX \int_{0}^{\Lambda} dr \frac{1}{\rho_A(r, X)} \leq C_1 \frac{1}{\Lambda}\),
2. \(\int_{-1}^{1} dX \int_{0}^{\Lambda} dr \left(\frac{1}{\rho_A(r, X)}\right)^2 \leq C_2 \frac{1}{\Lambda^{5/2}}\),
3. \(\int_{-1}^{1} dX \int_{0}^{\Lambda} dr \frac{1}{\rho_A(r, X)} \frac{1}{r+2} \leq C_3 \frac{\log \Lambda}{\Lambda^2}\),
4. \(\int_{-1}^{1} dX \int_{0}^{\Lambda} dr \left(\frac{1}{\rho_A(r, X)}\right)^2 (1 - X^2) \leq C_4 \frac{1}{\Lambda^{3}}\).

Using (1)–(4) we can prove that there exists a constant \(C > 0\) such that
\[
|b_j(\Lambda/m)| \leq C |\log(\Lambda/m)|^2, \quad j = 1, 4,
|b_j(\Lambda/m)| \leq C (\Lambda/m)^{1/2}, \quad j = 3, 5, 6.
\]

Hence there exists a constant \(C_{\text{max}}\) such that
\[
\lim_{\Lambda \to \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\text{max}}.
\]

Next we can show that there exists a positive constant \(\xi > 0\) such that
\[
\lim_{\Lambda \to \infty} \sqrt{\Lambda/m} \frac{d}{d(\Lambda/m)} b_2(\Lambda/m) > \xi,
\]
which implies that there exists a constant \(\xi'\) such that
\[
\xi' \leq \lim_{\Lambda \to \infty} b_2(\Lambda/m) \sqrt{\Lambda/m}.
\]

Thus we have
\[
C_{\text{min}} \leq \lim_{\Lambda \to \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\text{max}}.
\]
Remark 3.2 Theorem 3.1 may suggests $\gamma \geq 1/2$ uniformly in $e$ but $e \neq 0$.

Remark 3.3 (1) $a_2(\Lambda/m, \kappa/m) / \sqrt{\Lambda/m}$ converges to a nonnegative constant as $\Lambda \to \infty$. (2) By (3.1), we can define $a_2(\Lambda/m, 0)$ since $b_2(\Lambda/m)$ with $\kappa = 0$ are finite. Moreover $a_2(\Lambda/m, 0)$ also satisfies Theorem 3.1. (3) In the case of $\kappa = 0$, Chen [1] established that $H(0)$ has a ground state $\psi_0(0)$ but does not for $H(p)$ with $p \neq 0$.

4 Concluding remarks

The Pauli-Fierz Hamiltonian with the dipole approximation, $H_{\text{dip}}$, is defined by $H$ with $A_{\varphi}$ replaced by $1 \otimes A_{\varphi}(0)$, i.e.,

$$H_{\text{dip}} = \frac{1}{2m}(p \otimes 1 - e 1 \otimes A_{\varphi}(0))^2 + V \otimes 1 + 1 \otimes H_t.$$

Set $V \equiv 0$. Note that

$$[H_{\text{dip}}, P_{\text{total}}] \neq 0.$$

It is established in [5] that there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that

$$UH_{\text{dip}}U^{-1} = -\frac{1}{2(m + \delta m)} \Delta \otimes 1 + 1 \otimes H_t + e^2 G,$$

where

$$\delta m = m + e^2 \frac{2}{3} ||\hat{\varphi}/\omega||^2,$$

$$G = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 ||\hat{\varphi}/(t^2 + \omega^2)||^2}{m + (2e^2/3)||\hat{\varphi}/\sqrt{t^2 + \omega^2}||^2} dt.$$

Hence

$$[UH_{\text{dip}}U^{-1}, P_{\text{total}}] = 0.$$

Then we can define the effective mass $m_{\text{eff}}$ for $UH_{\text{dip}}U^{-1}$, and which is

$$m_{\text{eff}}/m = 1 + \alpha \frac{4}{3\pi} (\Lambda/m - \kappa/m).$$

Hence $\gamma = 1$, then the mass renormalization for $H_{\text{dip}}$ is not available.

References


