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On the Abhyankar's question for affine plane curves with one place at infinity

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1 Introduction

Let $C$ be an irreducible algebraic curve in complex affine plane $\mathbb{C}^2$. We say that $C$ has one place at infinity, if the closure of $C$ intersects with the \( \infty \)-line in $\mathbb{P}^2$ at only one point $P$ and $C$ is locally irreducible at that point $P$.

The problem of finding the canonical models of curves with one place at infinity under the polynomial transformations of the coordinates of $\mathbb{C}^2$ has been studied by many mathematicians since Suzuki [10] and Abhyankar–Moh [2] proved independently that the canonical model of $C$ is a line when $C$ is non-singular and simply connected.

Sathaye [8] introduce the Abhyankar's question for curves with one place at infinity and Sathaye–Stenerson [9] suggested a candidate of counter example for this question. However, they could not give the answer to the question since the root computation for a huge polynomial system was required.

We found a counter example for the Abhyankar's question using computer algebra system. In this report, we give the details.
2 Preliminaries

Let $C$ be a curve with one place at infinity defined by a polynomial equation $f(x, y) = 0$ in the complex affine plane $\mathbb{C}^2$. Assume that $\deg_x f = m$, $\deg_y f = n$ and $d = \gcd(m, n)$. The dual graph corresponding to the minimal resolution of the singularity of $C$ at infinity is the following [11]:

\[
\begin{array}{cccc}
E_{j_0} & \cdots & E_{i_2} & E_{i_h} \\
\circ & \cdots & \circ & \circ \\
\circ & \circ & \circ & \circ \\
E_{j_1} & E_{j_2} & E_{j_h} \\
\end{array}
\]

**Definition 1** ($\delta$-sequence) Let $f$ be the defining polynomial of a curve $C$ with one place at infinity. Let $\delta_k (0 \leq k \leq h)$ be the order of the pole of $f$ on $E_{j_k}$ in the above dual graph. We shall call the sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$ the $\delta$-sequence of $C$ (or of $f$).

We have the following fact since $\deg_x f = m$ and $\deg_y f = n$.

**Fact 1** $\delta_0 = n, \delta_1 = m$

We set $L_k$ for each $k (1 \leq k \leq h)$ like the following figure:

\[
\begin{array}{cccc}
E_{j_0} & \cdots & E_{i_2} & E_{i_h} \\
\circ & \cdots & \circ & \circ \\
\circ & \circ & \circ & \circ \\
E_{j_1} & E_{j_2} & E_{j_h} \\
\end{array}
\]

\[
\begin{array}{cccc}
L_1 & L_2 & L_h \\
\circ & \cdots & \circ & \circ \\
\circ & \circ & \circ & \circ \\
E_{j_1} & E_{j_2} & E_{j_h} \\
\end{array}
\]

**Definition 2** ($(p, q)$-sequence) Now, we assume that the weights of $L_k$ is of the following form:
We define the natural numbers $p_k, a_k, q_k, b_k$ satisfying

\[(p_k, a_k) = 1, \ (q_k, b_k) = 1, \ 0 < a_k < p_k, \ 0 < b_k < q_k,\]

\[\frac{p_k}{a_k} = m_1 - \frac{1}{m_2} - \frac{1}{m_3} - \ldots - \frac{1}{m_r}, \quad \frac{q_k}{b_k} = n_1 - \frac{1}{n_2} - \frac{1}{n_3} - \ldots - \frac{1}{n_s}.\]

We shall call the sequence \{$(p_1, q_1), (p_2, q_2), \ldots, (p_h, q_h)$\} the $(p, q)$-sequence of $C$ (or of $f$).

There are the following Abhyankar-Moh’s semigroup theorem and its converse theorem by Sathaye-Stenerson as results for $\delta$-sequence. We set $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

**Theorem 1 (Abhyankar-Moh [1, 3, 4])** Let $C$ be an affine plane curve with one place at infinity. Let \{\$\delta_0, \delta_1, \ldots, \delta_h\$\} be the $\delta$-sequence of $C$ and \{$(p_1, q_1), \ldots, (p_h, q_h)$\} be the $(p, q)$-sequence of $C$. We set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ ($1 \leq k \leq h + 1$). We have then,

(i) $q_k = d_k/d_{k+1}, \ d_{h+1} = 1$ ($1 \leq k \leq h$),

(ii) $d_{k+1}d_k = \begin{cases} \delta_1 & (k = 1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \leq k \leq h) \end{cases}$,

(iii) $q_k\delta_k \in \mathbb{N}\delta_0 + \mathbb{N}\delta_1 + \ldots + \mathbb{N}\delta_{k-1}$ ($1 \leq k \leq h$).

**Theorem 2 (Sathaye-Stenerson [9])** Let \{\$\delta_0, \delta_1, \ldots, \delta_h\$\} ($h \geq 1$) be the sequence of $h+1$ natural numbers. We set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ ($1 \leq k \leq h + 1$) and $q_k = d_k/d_{k+1}$ ($1 \leq k \leq h$). Furthermore, suppose that the following conditions are satisfied:
(1) \( \delta_0 < \delta_1 \),
(2) \( q_k \geq 2 (1 \leq k \leq h) \),
(3) \( d_{h+1} = 1 \),
(4) \( \delta_k < q_{k-1} \delta_{k-1} (2 \leq k \leq h) \),
(5) \( q_k \delta_k \in \mathbb{N} \delta_0 + \mathbb{N} \delta_1 + \cdots + \mathbb{N} \delta_{k-1} (1 \leq k \leq h) \).

Then, there exists a curve with one place at infinity of the \( \delta \)-sequence \( \{\delta_0, \delta_1, \ldots, \delta_h\} \).

Suzuki [11] gave an algebro-geometric proof of the above two theorem by the consideration of the resolution graph at infinity. Further, Suzuki gave an algorithm for mutual conversion of a dual graph and a \( \delta \)-sequence.

3 Construction of defining polynomials of curves

We shall assume that \( f(x, y) \) is monic in \( y \). We define approximate roots by Abhyankar’s definition.

**Definition 3** (approximate roots) Let \( f(x, y) \) be the defining polynomial, monic in \( y \), of a curve with one place at infinity. Let \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) be the \( \delta \)-sequence of \( f \). We set \( n = \deg_y f, d_k = \gcd \{\delta_0, \delta_1, \ldots, \delta_k-1\} \) and \( n_k = n/d_k (1 \leq k \leq h+1) \). Then, for each \( k (1 \leq k \leq h+1) \), a pair of polynomials \((g_k(x, y), \psi_k(x, y))\) satisfying the following conditions is uniquely determined:

(i) \( g_k \) is monic in \( y \) and \( \deg_y g_k = n_k \),
(ii) \( \deg_y \psi_k < n - n_k \),
(iii) \( f = g_k^{d_k} + \psi_k \).

We call this \( g_k \) the \( k \)-th approximate root of \( f \).

We can easily get the following fact from the definition of approximate roots.

**Fact 2** We have

\[
g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_k x^k, \quad g_{h+1} = f
\]

where \( c_k \in \mathbb{C}, p = \deg_x f/d, q = \deg_y f/d, d = \gcd \{\deg_x f, \deg_y f\} \) and \( \lfloor p/q \rfloor \) is the maximal integer \( \ell \) such that \( \ell \leq p/q \).
DEFINITION 4 (Abhyankar-Moh’s condition) We shall call the conditions (1) – (5) concerning \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) in Theorem 2 Abhyankar-Moh’s condition.

The following theorem gives normal forms of defining polynomials of curves with one place at infinity and the method of construction of their defining polynomials.

**Theorem 3 ([5])** Let \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) \((h \geq 1)\) be a sequence of natural numbers satisfying Abhyankar-Moh’s condition (see DEFINITION 4). Set \( d_k = \gcd \{\delta_0, \delta_1, \ldots, \delta_{k-1}\} \) \((1 \leq k \leq h + 1)\) and \( q_k = d_k/d_{k+1} \) \((1 \leq k \leq h)\).

1. We define \( g_k \) \((0 \leq k \leq h + 1)\) as follows:

\[
\begin{align*}
g_0 &= x, \\
g_1 &= y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbb{C}, \ p = \delta_1/d_2, \ q = \delta_0/d_2, \\
g_{i+1} &= g_i^{q_i} + a_{\overline{a}_0\overline{a}_1\ldots\overline{a}_{i-1}} g_0^{\overline{a}_0} g_1^{\overline{a}_1} \ldots g_{i-1}^{\overline{a}_{i-1}} \\
&\quad + \sum_{(a_0, a_1, \ldots, a_i) \in \Lambda_i} c_{a_0 a_1 \ldots a_i} g_0^{a_0} g_1^{a_1} \ldots g_i^{a_i},
\end{align*}
\]

where \((\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{i-1})\) is the sequence of \( i \) non-negative integers satisfying

\[
\sum_{j=0}^{i-1} \alpha_j \delta_j = q_i \delta_i, \quad \alpha_j < q_j \quad (0 < j < i)
\]

and

\[
\Lambda_i = \left\{ (a_0, a_1, \ldots, a_i) \in \mathbb{N}^{i+1} \mid \alpha_j < q_j \quad (0 < j < i), \ \alpha_i < q_i - 1, \ \sum_{j=0}^{i} \alpha_j \delta_j < q_i \delta_i \right\}.
\]

Then, \( g_0, g_1, \ldots, g_h \) are approximate roots of \( f = g_{h+1} \), and \( f \) is the defining polynomial, monic in \( y \), of a curve with one place at infinity of the \( \delta \)-sequence \( \{\delta_0, \delta_1, \ldots, \delta_h\} \).

2. The defining polynomial \( f \), monic in \( y \), of a curve with one place at infinity of the \( \delta \)-sequence \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) is obtained by the procedure of (1), and the values of parameters \( \{a_{\overline{a}_0\overline{a}_1\ldots\overline{a}_{i-1}}\}_{1 \leq i \leq h} \) and \( \{c_{a_0 a_1 \ldots a_i}\}_{0 \leq i \leq h} \) are uniquely determined for \( f \).
4 Abhyankar’s Question

**Definition 5** (planar semigroup) Let \( \{\delta_0, \delta_1, \ldots, \delta_h\} (h \geq 1) \) be a sequence of natural numbers satisfying Abhyankar-Moh’s condition. A semigroup generated by \( \{\delta_0, \delta_1, \ldots, \delta_h\} \) is said to be a planar semigroup.

**Definition 6** (polynomial curve) Let \( C \) be an algebraic curve defined by \( f(x, y) = 0 \), where \( f(x, y) \) is an irreducible polynomial in \( \mathbb{C}[x, y] \). We call \( C \) a polynomial curve, if \( C \) has a parametrisation \( x = x(t), y = y(t), \) where \( x(t) \) and \( y(t) \) are polynomials in \( \mathbb{C}[t] \).

**Abhyankar’s Question**: Let \( \Omega \) be a planar semigroup. Is there a polynomial curve with \( \delta \)-sequence generating \( \Omega \)?

Moh [6] showed that there is no polynomial curve with \( \delta \)-sequence \( \{6, 8, 3\} \). But there is a polynomial curve \( (x, y) = (t^3, t^8) \) with \( \delta \)-sequence \( \{3, 8\} \) which generates the same semigroup as above. Sathaye–Stenerson [9] proved that the semigroup generated by \( \{6, 22, 17\} \) has no other \( \delta \)-sequence generating the same semigroup, and proposed the following conjecture for this question.

**Sathaye–Stenerson’s Conjecture**: There is no polynomial curve having the \( \delta \)-sequence \( \{6, 22, 17\} \).

By Theorem 3, the defining polynomial of the curve with one place at infinity of the \( \delta \)-sequence \( \{6, 22, 17\} \) as follows:

\[
\begin{align*}
f &= (g_2^2 + a_{2,1}x^2g_1) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 \\
&\quad + c_{1,1,0}xg_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0}
\end{align*}
\]

where

\[
\begin{align*}
g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0, \\
g_2 &= (g_1^3 + a_{11}x^{11}) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\
&\quad + (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\
&\quad + (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}.
\end{align*}
\]

Since \( C \) has one place at infinity and genus zero if and only if \( C \) has polynomial parametrization (Abhyankar), \( \{6, 22, 17\} \) is a counter example if it can be shown that the above type curve does not include a polynomial curve.
5 Approach by using a computer algebra system

We assume that $C$ is a polynomial curve and has the $\delta$-sequence $\{6, 22, 17\}$. Therefore $C$ has the following polynomial parametrization:

\[
\begin{align*}
x &= t^5 + a_1 t^4 + a_2 t^3 + a_3 t^2 + a_4 t + a_5 + a_6 \\
y &= t^{22} + b_1 t^{21} + b_2 t^{20} + b_3 t^{19} + \cdots + b_{21} t + b_{22}
\end{align*}
\]

It follows that $\deg g_2(x(t), y(t)) = 17$ from the form of $f$ and $g_2$ in the previous section. We can get the polynomial system $I$ with 11 variables and 17 polynomials after eliminating variables from the coefficients of all terms of $t$-degree more than 18 in $g_2(x(t), y(t))$.

$\{6, 22, 17\}$ is a counter example of Abhyankar's question if $I$ does not have a root. For such a huge polynomial system it is suitable to compute the Gröbner basis of the ideal. However, it was impossible to compute the Gröbner basis of $I$ even if using a computer with 8GB memory.

We classified $\delta$-sequences with genus $\leq 50$ into groups which generate the same semigroup. Furthermore, we listed $\delta$-sequences with the following three properties: (i) There is no other $\delta$-sequence which generates the same semigroup. (ii) The number of generators is 3. (iii) $k$-number $\geq -1$. Then, we obtained $\{6, 15, 4\}, \{4, 14, 9\}, \{6, 15, 7\}, \{6, 21, 4\}, \cdots$. The Gröbner basis computations for the polynomial systems corresponding to these $\delta$-sequences showed that $\{6, 21, 4\}$ was a counter example of Abhyankar's question.

The defining polynomial of the curve with one place at infinity of the $\delta$-sequence $\{6, 21, 4\}$ as follows:

\[
f = g_2^3 + a_{2,0} x^2 + c_{1,0,1} x g_2 + c_{1,0,0} x + c_{0,0,1} g_2 + c_{0,0,0}
\]

where

\[
\begin{align*}
g_2 &= g_1^2 + a_7 x^7 + c_{6,0} x^6 + c_{5,0} x^5 + c_{4,0} x^4 + c_{3,0} x^3 \\
&\quad + c_{2,0} x^2 + c_{1,0} x + c_{0,0} \\
g_1 &= y + c_3 x^3 + c_2 x^2 + c_1 x + c_0
\end{align*}
\]

Let the following be the polynomial parametrization of the polynomial curve with $\delta$-sequence $\{6, 21, 4\}$:

\[
\begin{align*}
x &= t^6 + a_1 t^5 + a_2 t^4 + a_3 t^3 + a_4 t^2 + a_5 t + a_6 \\
y &= t^{21} + b_1 t^{20} + b_2 t^{19} + b_3 t^{18} + \cdots + b_{20} t + b_{21}
\end{align*}
\]
By the same operation as the case of \{6, 22, 17\} we can get the polynomial system \( J \) with 7 variables \( \{a_2, a_3, a_4, a_5, a_6, b_{12}, b_{18}\} \) and 13 polynomials from \( \deg_t g_2(x(t), y(t)) = 4 \).

We used the total degree reverse lexicographic ordering (DRL) with \( a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_6 \succ b_{12} \succ b_{18} \) to the Gröbner basis computation. CPU time for the computation is 3 hours 40 minutes and the required memory is 850MB. The computer is a PC AthlonMP 2200+ with 4GB memory. The computer algebra system is Risa/Asir [7] on FreeBSD 4.7.

The obtained Gröbner basis \( G \) of \( J \) was not \{1\}. However, the normal form of the coefficient \( p \) of the term with \( t \)-degree = 4 in \( g_2(x(t), y(t)) \) with respect to \( G \) is 0. This shows that \( p \in J \). Thus, we get \( \deg_t g_2(x(t), y(t)) < 4 \). Since this is contradictory for \( \deg_t g_2(x(t), y(t)) = 4 \), there is no polynomial curve with \( \delta \)-sequence \{6, 21, 4\}.

References


